

Green's Functions in Physics

Version 1

M. Baker, S. Sutlief

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Preface

This manuscript is based on lectures given by Marshall Baker for a class on Mathematical Methods in Physics at the University of Washington in 1988. The subject of the lectures was Green's function techniques in Physics. All the members of the class had completed the equivalent of the first three and a half years of the undergraduate physics program, although some had significantly more background. The class was a preparation for graduate study in physics.

These notes develop Green's function techniques for both single and multiple dimension problems, and then apply these techniques to solving the wave equation, the heat equation, and the scattering problem. Many other mathematical techniques are also discussed.

To read this manuscript it is best to have Arfken's book handy for the mathematics details and Fetter and Walecka's book handy for the physics details. There are other good books on Green's functions available, but none of them are geared for same background as assumed here. The two volume set by Stakgold is particularly useful. For a strictly mathematical discussion, the book by Dennery is good.

Here are some notes and warnings about this revision:

- *Text* This text is an amplification of lecture notes taken of the Physics 425-426 sequence. Some sections are still a bit rough. Be alert for errors and omissions.
- *List of Symbols* A listing of mostly all the variables used is included. Be warned that many symbols are created ad hoc, and thus are only used in a particular section.
- *Bibliography* The bibliography includes those books which have been useful to Steve Sutlief in creating this manuscript, and were

not necessarily used for the development of the original lectures. Books marked with an asterisk are more supplemental. Comments on the books listed are given above.

- *Index* The index was composed by skimming through the text and picking out places where ideas were introduced or elaborated upon. No attempt was made to locate all relevant discussions for each idea.

A Note About Copying:

These notes are in a state of rapid transition and are provided so as to be of benefit to those who have recently taken the class. Therefore, please do not photocopy these notes.

Contacting the Authors:

A list of phone numbers and email addresses will be maintained of those who wish to be notified when revisions become available. If you would like to be on this list, please send email to

sutlief@u.washington.edu

before 1996. Otherwise, call Marshall Baker at 206-543-2898.

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This manuscript benefits greatly from the excellent set of notes taken by Steve Griffies. Richard Horn contributed many corrections and suggestions. Special thanks go to the students of Physics 425-426 at the University of Washington during 1988 and 1993.

This first revision contains corrections only. No additional material has been added since Version 0.

Steve Sutlief
Seattle, Washington
16 June, 1993
4 January, 1994

Chapter 1

The Vibrating String

Chapter Goals:

- Construct the wave equation for a string by identifying forces and using Newton's second law.
- Determine boundary conditions appropriate for a closed string, an open string, and an elastically bound string.
- Determine the wave equation for a string subject to an external force with harmonic time dependence.

4 Jan p1
p1prv.yr.

The central topic under consideration is the branch of differential equation theory containing boundary value problems. First we look at an example of the application of Newton's second law to small vibrations: transverse vibrations on a string. Physical problems such as this and those involving sound, surface waves, heat conduction, electromagnetic waves, and gravitational waves, for example, can be solved using the mathematical theory of boundary value problems.

Consider the problem of a string embedded in a medium with a restoring force $V(x)$ and an external force $F(x, t)$. This problem covers most of the physical interpretations of small vibrations. In this chapter we will investigate the mathematics of this problem by determining the equations of motion.

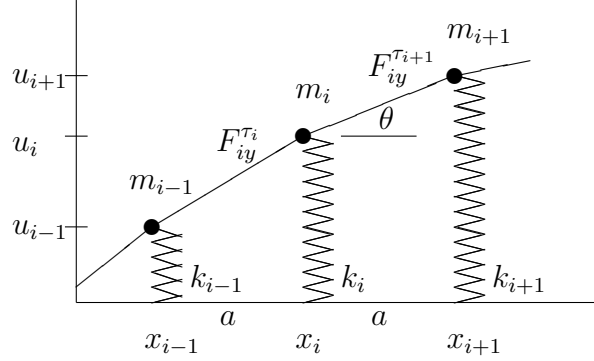


Figure 1.1: A string with mass points attached to springs.

1.1 The String

We consider a massless string with equidistant mass points attached. In the case of a string, we shall see (in chapter 3) that the Green's function corresponds to an impulsive force and is represented by a complete set of functions. Consider N mass points of mass m_i attached to a massless string, which has a tension τ between mass points. An elastic force at each mass point is represented by a spring. This problem is illustrated in figure 1.1 We want to find the equations of motion for transverse vibrations of the string.

1.1.1 Forces on the String

For the massless vibrating string, there are three forces which are included in the equation of motion. These forces are the tension force, elastic force, and external force.

Tension Force

For each mass point there are two force contributions due to the tension on the string. We call τ_i the tension on the segment between m_{i-1} and m_i , u_i the vertical displacement of the i th mass point, and a the horizontal displacement between mass points. Since we are considering transverse vibrations (in the u -direction) , we want to know the tension

pr:N1
pr:mi1
pr:tau1
fig1.1
pr:eom1

4 Jan p2
pr:tension1
pr:ui1
pr:a1
pr:transvib1

force in the u -direction, which is $\tau_{i+1} \sin \theta$. From the figure we see that $\theta \approx (u_{i+1} - u_i)/a$ for small angles and we can thus write

$$F_{iy}^{\tau_{i+1}} = \tau_{i+1} \frac{(u_{i+1} - u_i)}{a}$$

and

$$F_{iy}^{\tau_i} = -\tau_i \frac{(u_i - u_{i-1})}{a}.$$

Note that the equations agree with dimensional analysis:

$$\begin{aligned} F_{iy}^{\tau_i} &= \dim(m \cdot l/t^2), & \tau_i &= \dim(m \cdot l/t^2), \\ u_i &= \dim(l), & \text{and} & \quad a = \dim(l). \end{aligned}$$

Elastic Force

We add an elastic force with spring constant k_i :

$$F_i^{\text{elastic}} = -k_i u_i,$$

where $\dim(k_i) = (m/t^2)$. This situation can be visualized by imagining vertical springs attached to each mass point, as depicted in figure 1.1. A small value of k_i corresponds to an elastic spring, while a large value of k_i corresponds to a rigid spring.

External Force

We add the external force F_i^{ext} . This force depends on the nature of the physical problem under consideration. For example, it may be a transverse force at the end points.

1.1.2 Equations of Motion for a Massless String

The problem thus far has concerned a massless string with mass points attached. By summing the above forces and applying Newton's second law, we have

$$F_{\text{tot}} = \tau_{i+1} \frac{(u_{i+1} - u_i)}{a} - \tau_i \frac{(u_i - u_{i-1})}{a} - k_i u_i + F_i^{\text{ext}} = m_i \frac{d^2}{dt^2} u_i. \quad (1.1)$$

This gives us N coupled inhomogeneous linear ordinary differential equations where each u_i is a function of time. In the case that F_i^{ext} is zero we have free vibration, otherwise we have forced vibration.

1.1.3 Equations of Motion for a Massive String

4 Jan p3

For a string with continuous mass density, the equidistant mass points on the string are replaced by a continuum. First we take a , the separation distance between mass points, to be small and redefine it as $a = \Delta x$. We correspondingly write $u_i - u_{i-1} = \Delta u$. This allows us to write

pr:deltax1
pr:deltau1

$$\frac{(u_i - u_{i-1})}{a} = \left(\frac{\Delta u}{\Delta x} \right)_i. \quad (1.2)$$

The equations of motion become (after dividing both sides by Δx)

$$\frac{1}{\Delta x} \left[\tau_{i+1} \left(\frac{\Delta u}{\Delta x} \right)_{i+1} - \tau_i \left(\frac{\Delta u}{\Delta x} \right)_i \right] - \frac{k_i}{\Delta x} u_i + \frac{F_i^{\text{ext}}}{\Delta x} = \frac{m_i}{\Delta x} \frac{d^2 u_i}{dt^2}. \quad (1.3)$$

eq1deltf

In the limit we take $a \rightarrow 0$, $N \rightarrow \infty$, and define their product to be

$$\lim_{\substack{a \rightarrow 0 \\ N \rightarrow \infty}} Na \equiv L. \quad (1.4)$$

pr:signmax1

The limiting case allows us to redefine the terms of the equations of motion as follows:

$$\begin{aligned} m_i \rightarrow 0 & \quad \frac{m_i}{\Delta x} \rightarrow \sigma(x_i) \equiv \frac{\text{mass}}{\text{length}} = \text{mass density}; \\ k_i \rightarrow 0 & \quad \frac{k_i}{\Delta x} \rightarrow V(x_i) = \text{coefficient of elasticity of the media}; \\ F_i^{\text{ext}} \rightarrow 0 & \quad \frac{F_i^{\text{ext}}}{\Delta x} = \left(\frac{m_i}{\Delta x} \cdot \frac{F_i^{\text{ext}}}{m_i} \right) \rightarrow \sigma(x_i) f(x_i) \end{aligned} \quad (1.5)$$

where

$$f(x_i) = \frac{F_i^{\text{ext}}}{m_i} = \frac{\text{external force}}{\text{mass}}. \quad (1.6)$$

Since

$$\begin{aligned} x_i &= x \\ x_{i-1} &= x - \Delta x \\ x_{i+1} &= x + \Delta x \end{aligned}$$

pr:x1

we have

$$\left(\frac{\Delta u}{\Delta x} \right)_i = \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \rightarrow \frac{\partial u(x, t)}{\partial x} \quad (1.7)$$

so that

$$\begin{aligned} & \frac{1}{\Delta x} \left[\tau_{i+1} \left(\frac{\Delta u}{\Delta x} \right)_{i+1} - \tau_i \left(\frac{\Delta u}{\Delta x} \right)_i \right] \\ &= \frac{1}{\Delta x} \left[\tau(x + \Delta x) \frac{\partial u(x + \Delta x)}{\partial x} - \tau(x) \frac{\partial u(x)}{\partial x} \right] \\ &= \frac{\partial}{\partial x} \left[\tau(x) \frac{\partial u}{\partial x} \right]. \end{aligned} \quad (1.8)$$

This allows us to write 1.3 as

4 Jan p4

$$\frac{\partial}{\partial x} \left[\tau(x) \frac{\partial u}{\partial x} \right] - V(x)u + \sigma(x)f(x, t) = \sigma(x) \frac{\partial^2 u}{\partial t^2}. \quad (1.9)$$

This is a partial differential equation. We will look at this problem in detail in the following chapters. Note that the first term is net tension force over dx . eq1diff
pr:pde1

1.2 The Linear Operator Form

We define the linear operator L_0 by the equation

pr:LinOp1

$$L_0 \equiv -\frac{\partial}{\partial x} \left(\tau(x) \frac{\partial}{\partial x} \right) + V(x). \quad (1.10)$$

We can now write equation (1.9) as

eq1LinOp

$$\left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] u(x, t) = \sigma(x)f(x, t) \quad \text{on } a < x < b. \quad (1.11)$$

This is an inhomogeneous equation with an external force term. Note that each term in this equation has units of m/t^2 . Integrating this equation over the length of the string gives the total force on the string. eq1waveone

1.3 Boundary Conditions

To obtain a unique solution for the differential equation, we must place restrictive conditions on it. In this case we place conditions on the ends of the string. Either the string is tied together (i.e. closed), or its ends are left apart (open). pr:bc1

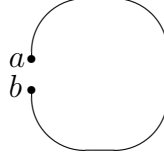


Figure 1.2: A closed string, where a and b are connected.

1.3.1 Case 1: A Closed String

pr:ClStr1
pr:a2
fig1loop
pr:pbcl

A closed string has its endpoints a and b connected. This case is illustrated in figure 2. This is the *periodic boundary condition* for a closed string. A closed string must satisfy the following equations:

$$u(a, t) = u(b, t) \quad (1.12)$$

eq1pbc1

which is the condition that the ends meet, and

$$\left. \frac{\partial u(x, t)}{\partial x} \right|_{x=a} = \left. \frac{\partial u(x, t)}{\partial x} \right|_{x=b} \quad (1.13)$$

eq1pbc2

which is the condition that the ends have the same declination (i.e., the string must be smooth across the end points).

1.3.2 Case 2: An Open String

sec1-c2
4 Jan p5
pr:ebcl
pr:OpStr1

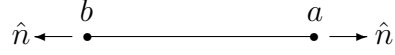
For an elastically bound open string we have the boundary condition that the total force must vanish at the end points. Thus, by multiplying equation 1.3 by Δx and setting the right hand side equal to zero, we have the equation

$$\tau_a \left. \frac{\partial u(x, t)}{\partial x} \right|_{x=a} - k_a u(a, t) + F_a(t) = 0.$$

pr:ha1

The homogeneous terms of this equation are $\tau_a \frac{\partial u}{\partial x}|_{x=a}$ and $k_a u(a, t)$, and the inhomogeneous term is $F_a(t)$. The term $k_a u(a)$ describes how the string is bound. We now define

$$h_a(t) \equiv \frac{F_a}{\tau_a} \quad \text{and} \quad \kappa_a \equiv \frac{k_a}{\tau_a}.$$

Figure 1.3: An open string, where the endpoints a and b are free.

The term $h_a(t)$ is the effective force and κ_a is the effective spring constant.

$$-\frac{\partial u}{\partial x} + \kappa_a u(x) = h_a(t) \quad \text{for } x = a. \quad (1.14)$$

We also define the outward normal, \hat{n} , as shown in figure 1.3. This allows us to write 1.14 as

$$\hat{n} \cdot \nabla u(x) + \kappa_a u(x) = h_a(t) \quad \text{for } x = a.$$

The boundary condition at b can be similarly defined:

$$\frac{\partial u}{\partial x} + \kappa_b u(x) = h_b(t) \quad \text{for } x = b,$$

where

$$h_b(t) \equiv \frac{F_b}{\tau_b} \quad \text{and} \quad \kappa_b \equiv \frac{k_b}{\tau_b}.$$

For a more compact notation, consider points a and b to be elements of the “surface” of the one dimensional string, $S = \{a, b\}$. This gives us

$$\hat{n}_S \nabla u(x) + \kappa_S u(x) = h_S(t) \quad \text{for } x \text{ on } S, \text{ for all } t. \quad (1.15)$$

In this case $\hat{n}_a = -\vec{l}_x$ and $\hat{n}_b = \vec{l}_x$.

1.3.3 Limiting Cases

It is also worthwhile to consider the limiting cases for an elastically bound string. These cases may be arrived at by varying κ_a and κ_b . The terms κ_a and κ_b signify how rigidly the string’s endpoints are bound. The two limiting cases of equation 1.14 are as follows:

$$\kappa_a \rightarrow 0 \quad -\frac{\partial u}{\partial x}\bigg|_{x=a} = h_a(t) \quad (1.16)$$

$$\kappa_a \rightarrow \infty \quad u(x, t)|_{x=a} = h_a/\kappa_a = F_a/k_a. \quad (1.17)$$

The boundary condition $\kappa_a \rightarrow 0$ corresponds to an elastic media, and is called the Neumann boundary condition. The case $\kappa_a \rightarrow \infty$ corresponds to a rigid medium, and is called the Dirichlet boundary condition.

If $h_S(t) = 0$ in equation 1.15, so that

$$[\hat{n}_S \cdot \nabla + \kappa_S]u(x, t) = h_S(t) = 0 \quad \text{for } x \text{ on } S, \quad (1.18)$$

then the boundary conditions are called *regular boundary conditions*. Regular boundary conditions are either

1. $u(a, t) = u(b, t)$, $\frac{d}{dx}u(a, t) = \frac{d}{dx}u(b, t)$ (periodic), or
2. $[\hat{n}_S \cdot \nabla + \kappa_S]u(x, t) = 0$ for x on S .

Thus regular boundary conditions correspond to the case in which there is no external force on the end points.

1.3.4 Initial Conditions

The complete description of the problem also requires information about the string at some reference point in time:

$$u(x, t)|_{t=0} = u_0(x) \quad \text{for } a < x < b \quad (1.19)$$

and

$$\frac{\partial}{\partial t}u(x, t)|_{t=0} = u_1(x) \quad \text{for } a < x < b. \quad (1.20)$$

Here we claim that it is sufficient to know the position and velocity of the string at some point in time.

1.4 Special Cases

We now consider two singular boundary conditions and a boundary condition leading to the Helmholtz equation. The conditions first two cases will ensure that the right-hand side of Green's second identity (introduced in chapter 2) vanishes. This is necessary for a physical system.

1.4.1 No Tension at Boundary

For the case in which $\tau(a) = 0$ and the regular boundary conditions hold, the condition that $u(a)$ be finite is necessary. This is enough to ensure that the right hand side of Green's second identity is zero.

1.4.2 Semi-infinite String

In the case that $a \rightarrow -\infty$, we require that $u(x)$ have a finite limit as $x \rightarrow -\infty$. Similarly, if $b \rightarrow \infty$, we require that $u(x)$ have a finite limit as $x \rightarrow \infty$. If both $a \rightarrow -\infty$ and $b \rightarrow \infty$, we require that $u(x)$ have finite limits as either $x \rightarrow -\infty$ or $x \rightarrow \infty$.

1.4.3 Oscillatory External Force

sec1helm

In the case in which there are no forces at the boundary we have

$$h_a = h_b = 0. \quad (1.21)$$

The terms h_a, h_b are extra forces on the boundaries. Thus the condition of no forces on the boundary does not imply that the internal forces are zero. We now treat the case where the interior force is oscillatory and write

pr:omega1

$$f(x, t) = f(x)e^{-i\omega t}. \quad (1.22)$$

In this case the physical solution will be

$$\operatorname{Re} f(x, t) = f(x) \cos \omega t. \quad (1.23)$$

We look for steady state solutions of the form

pr:sss1

$$u(x, t) = e^{-i\omega t} u(x) \quad \text{for all } t. \quad (1.24)$$

This gives us the equation

$$\left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] e^{-i\omega t} u(x) = \sigma(x) f(x) e^{-i\omega t}. \quad (1.25)$$

If $u(x, \omega)$ satisfies the equation

$$[L_0 - \omega^2 \sigma(x)] u(x) = \sigma(x) f(x) \quad \text{with R.B.C. on } u(x) \quad (1.26)$$

(the Helmholtz equation), then a solution exists. We will solve this *eq1helm*
equation in chapter 3. *pr:Helm1*

1.5 Summary

In this chapter the equations of motion have been derived for the small oscillation problem. Appropriate forms of the boundary conditions and initial conditions have been given.

The general string problem with external forces is mathematically the same as the small oscillation (vibration) problem, which uses vectors and matrices. Let $u_i = u(x_i)$ be the amplitude of the string at the point x_i . For the discrete case we have N component vectors $u_i = u(x_i)$, and for the continuum case we have a continuous function $u(x)$. These considerations outline the most general problem.

The main results for this chapter are:

1. The equation of motion for a string is

$$\left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] u(x, t) = \sigma(x) f(x, t) \quad \text{on } a < x < b$$

where

$$L_0 u = \left[-\frac{\partial}{\partial x} \left(\tau(x) \frac{\partial}{\partial x} \right) + V(x) \right] u.$$

2. *Regular boundary conditions* refer to the boundary conditions for either

- (a) a closed string:

$$u(a, t) = u(b, t) \quad (\text{continuous})$$

$$\left. \frac{\partial u(a, t)}{\partial x} \right|_{x=a} = \left. \frac{\partial u(b, t)}{\partial x} \right|_{x=b} \quad (\text{no bends})$$

or

- (b) an open string:

$$[\hat{n}_S \cdot \nabla + \kappa_S] u(x, t) = h_S(t) = 0 \quad x \text{ on } S, \text{ all } t.$$

3. The initial conditions are given by the equations

$$u(x, t)|_{t=0} = u_0(x) \quad \text{for } a < x < b \quad (1.27)$$

and

$$\left. \frac{\partial}{\partial t} u(x, t) \right|_{t=0} = u_1(x) \quad \text{for } a < x < b. \quad (1.28)$$

4. The Helmholtz equation is

$$[L_0 - \omega^2 \sigma(x)]u(x) = \sigma(x)f(x).$$

1.6 References

See any book which derives the wave equation, such as [Fetter80, p120ff], [Griffiths81, p297], [Halliday78, pA5].

A more thorough definition of regular boundary conditions may be found in [Stakgold67a, p268ff].

Chapter 2

Green's Identities

Chapter Goals:

- Derive Green's first and second identities.
- Show that for regular boundary conditions, the linear operator is hermitian.

In this chapter, appropriate tools and relations are developed to solve the equation of motion for a string developed in the previous chapter. In order to solve the equations, we will want the function $u(x)$ to take on complex values. We also need the notion of an inner product. The inner product of S and u is defined as

note
pr:InProd1
pr:S2

$$\langle S, u \rangle = \begin{cases} \sum_{i=1}^n S_i^* u_i & \text{for the discrete case} \\ \int_a^b dx S^*(x) u(x) & \text{for the continuous case.} \end{cases} \quad (2.1)$$

In the uses of the inner product which will be encountered here, for the continuum case, one of the variables S or u will be a length (amplitude of the string), and the other will be a force per unit length. Thus the inner product will have units of force times length, which is work.

eq2.2

2.1 Green's 1st and 2nd Identities

6 Jan p2.4

In the definition of the inner product we make the substitution of L_0u for u , where

$$L_0u(x) \equiv \left[-\frac{d}{dx} \left(\tau(x) \frac{d}{dx} \right) + V(x) \right] u(x). \quad (2.2)$$

eq2.3

This substitution gives us

$$\begin{aligned} \langle S, L_0u \rangle &= \int_a^b dx S^*(x) \left[-\frac{d}{dx} \left(\tau(x) \frac{d}{dx} \right) + V(x) \right] u(x) \\ &= - \int_a^b dx S^*(x) \left(-\frac{d}{dx} \left(\tau(x) \frac{d}{dx} u \right) \right) \\ &\quad + \int_a^b dx S^*(x) V(x) u(x). \end{aligned}$$

We now integrate twice by parts ($\int \bar{u} d\bar{v} = \bar{u}\bar{v} - \int \bar{v} d\bar{u}$), letting

$$\bar{u} = S^*(x) \implies d\bar{u} = dS^*(x) = dx \frac{dS^*(x)}{dx}$$

and

$$d\bar{v} = dx \frac{d}{dx} \left(\tau(x) \frac{d}{dx} u \right) = d \left(\tau(x) \frac{d}{dx} u \right) \implies \bar{v} = \tau(x) \frac{d}{dx} u$$

so that

$$\begin{aligned} \langle S, L_0u \rangle &= - \left[S^*(x) \tau(x) \left(\frac{d}{dx} u(x) \right) \right]_a^b \\ &\quad + \int_a^b dx \frac{dS^*}{dx} \tau(x) \left(\frac{d}{dx} u(x) \right) + \int_a^b dx S^*(x) V(x) u(x) \\ &= - \left[S^*(x) \tau(x) \frac{d}{dx} u(x) \right]_a^b \\ &\quad + \int_a^b dx \left[\left(\frac{d}{dx} S^* \right) \tau(x) \frac{d}{dx} u(x) + S^*(x) V(x) u(x) \right]. \end{aligned}$$

Note that the final integrand is symmetric in terms of $S^*(x)$ and $u(x)$.

This is *Green's First Identity*.

pr:G1Id1

$$\begin{aligned} \langle S, L_0 u \rangle &= - \left[S^*(x) \tau(x) \frac{d}{dx} u(x) \right]_a^b \\ &\quad + \int_a^b dx \left[\left(\frac{d}{dx} S^* \right) \tau(x) \frac{d}{dx} u(x) + S^*(x) V(x) u(x) \right]. \end{aligned} \quad (2.3)$$

Now interchange S^* and u to get

eq2G1Id

$$\begin{aligned} \langle u, L_0 S \rangle^* &= \langle L_0 S, u \rangle \\ &= - \left[u(x) \tau(x) \frac{d}{dx} S^*(x) \right]_a^b \\ &\quad + \int_a^b dx \left[\left(\frac{d}{dx} u \right) \tau(x) \frac{d}{dx} S^*(x) + u(x) V(x) S^*(x) \right]. \end{aligned} \quad (2.4)$$

When the difference of equations 2.3 and 2.4 is taken, the symmetric terms cancel. This is *Green's Second Identity*.

eq2preG2Id
pr:G2Id1

$$\langle S, L_0 u \rangle - \langle L_0 S, u \rangle = \left[\tau(x) \left[u(x) \frac{d}{dx} S^*(x) - S^*(x) \frac{d}{dx} u(x) \right] \right]_a^b. \quad (2.5)$$

In the literature, the expressions for the Green's identities take $\tau = -1$ and $V = 0$ in the operator L_0 . Furthermore, the expressions here are for one dimension, while the multidimensional generalization is given in section 8.4.1.

eq2G2Id

2.2 Using G.I. #2 to Satisfy R.B.C.

6 Jan p2.5

The regular boundary conditions for a string (either equations 1.12 and 1.13 or equation 1.18) can simplify Green's 2nd Identity. If S and u correspond to physical quantities, they must satisfy RBC. We will verify this statement for two special cases: the closed string and the open string.

2.2.1 The Closed String

For a closed string we have (from equations 1.12 and 1.13)

$$u(a, t) = u(b, t), \quad S^*(a, t) = S^*(b, t),$$

$$\tau(a) = \tau(b), \quad \left. \frac{d}{dx} S^* \right|_{x=a} = \left. \frac{d}{dx} S^* \right|_{x=b}, \quad \left. \frac{d}{dx} u \right|_{x=a} = \left. \frac{d}{dx} u \right|_{x=b}.$$

By plugging these equalities into Green's second identity, we find that

$$\langle S, L_0 u \rangle = \langle L_0 S, u \rangle. \quad (2.6)$$

eq2twox

2.2.2 The Open String

For an open string we have

$$\begin{aligned} -\frac{\partial u}{\partial x} + K_a u &= 0 & \text{for } x = a, \\ -\frac{\partial S^*}{\partial x} + K_a S^* &= 0 & \text{for } x = a, \\ \frac{\partial u}{\partial x} + K_b u &= 0 & \text{for } x = b, \\ \frac{\partial S^*}{\partial x} + K_b S^* &= 0 & \text{for } x = b. \end{aligned} \quad (2.7)$$

eq21osbc

These are the conditions for RBC from equation 1.14. Plugging these expressions into Green's second identity gives

$$\left|_a \tau(x) \left[u \frac{dS^*}{dx} - S^* \frac{du}{dx} \right] = \tau(a) [u K_a S^* - S^* K_a u] = 0$$

and

$$\left|_b^b \tau(x) \left[u \frac{dS^*}{dx} - S^* \frac{du}{dx} \right] = \tau(b) [u K_b S^* - S^* K_b u] = 0.$$

Thus from equation 2.5 we find that

$$\langle S, L_0 u \rangle = \langle L_0 S, u \rangle, \quad (2.8)$$

eq2twox2

just as in equation 2.6 for a closed string.

2.2.3 A Note on Hermitian Operators

The equation $\langle S, L_0 u \rangle = \langle L_0 S, u \rangle$, which we have found to hold for both a closed string and an open string, is the criterion for L_0 to be a *Hermitian operator*. By using the definition 2.1, this expression can be rewritten as

$$\langle S, L_0 u \rangle = \langle u, L_0 S \rangle^*. \quad (2.9)$$

Hermitian operators are generally generated by nondissipative physical problems. Thus Hermitian operators with Regular Boundary Conditions are generated by nondissipative mechanical systems. In a dissipative system, the acceleration cannot be completely specified by the position and velocity, because of additional factors such as heat, friction, and/or other phenomena.

2.3 Another Boundary Condition

6 Jan p2.6

If the ends of an open string are free of horizontal forces, the tension at the end points must be zero. Since

$$\lim_{x \rightarrow a, b} \tau(x) = 0$$

we have

$$\lim_{x \rightarrow a, b} \tau(x) u(x) \frac{\partial}{\partial x} S^*(x) = 0$$

and

$$\lim_{x \rightarrow a, b} \tau(x) S^*(x) \frac{\partial}{\partial x} u(x) = 0.$$

In the preceding equations, the abbreviated notation $\lim_{x \rightarrow a, b}$ is introduced to represent either the limit as x approaches the endpoint a or the limit as x approaches the endpoint b . These equations allow us to rewrite Green's second identity (equation 2.5) as

$$\langle S, L_0 u \rangle = \langle L_0 S, u \rangle \quad (2.10)$$

for the case of zero tension on the end points This is another way of getting at the result in equation 2.8 for the special case of free ends.

2.4 Physical Interpretations of the G.I.s

sec2.4

Certain qualities of the Green's Identities correspond to physical situations and constraints.

2.4.1 The Physics of Green's 2nd Identity

6 Jan p2.6

The right hand side of Green's 2nd Identity will always vanish for physically realizable systems. Thus L_0 is Hermitian for any physically realizable system.

We could extend the definition of regular boundary conditions by letting them be those in which the right-hand side of Green's second identity vanishes. This would allow us to include a wider class of problems, including singular boundary conditions, domains, and operators. This will be necessary to treat Bessel's equation. For now, however, we only consider problems whose boundary conditions are periodic or of the form of equation 1.18.

2.4.2 A Note on Potential Energy

The potential energy of an element dx of the string has two contributions. One is the "spring" potential energy $\frac{1}{2}V(x)(u(x))^2$ (c.f., $\frac{1}{2}kx^2$ in $U = -\int F dx = -\int (-kx) dx = \frac{1}{2}kx^2$ [Halliday76, p141]). The other is the "tension" potential energy, which comes from the tension force in section 1.1.3, $dF = \frac{\partial}{\partial x}[\tau(x)\frac{\partial}{\partial x}u(x)]dx$, and thus U_{tension} is

$$U = - \int \frac{\partial}{\partial x} \left(\tau \frac{\partial u}{\partial x} \right) dx,$$

so

$$\frac{dU}{dt} = - \frac{d}{dt} \int \frac{\partial}{\partial x} \left(\tau \frac{\partial u}{\partial x} \right) dx = - \int_x \frac{\partial}{\partial x} \left(\tau \frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial t} \right) dx,$$

and so the change in potential energy in a time interval dt is

$$\begin{aligned} U dt &= - \int_a^b \frac{\partial}{\partial x} \left(\tau \frac{\partial u}{\partial x} \right) \left(\frac{\partial u}{\partial t} \right) dt dx \\ &\stackrel{\text{parts}}{=} \int_a^b \left(\tau \frac{\partial u}{\partial x} \right) \frac{\partial}{\partial t} \frac{\partial u}{\partial x} dt dx - \left[\left(\tau \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dt \right]_a^b \end{aligned}$$

$$\begin{aligned}
&= \int_a^b \left(\tau \frac{\partial u}{\partial x} \right) \frac{\partial}{\partial t} \frac{\partial u}{\partial x} dt dx \\
&= \left[\frac{\partial}{\partial t} \int_a^b \frac{1}{2} \tau \left(\frac{\partial u}{\partial x} \right)^2 dx \right]_t^{t+dt}.
\end{aligned}$$

The second term in the second equality vanishes. We may now sum the differentials of U in time to obtain the potential energy:

$$U = \int_{t'=0}^t U dt = \left[\int_a^b \frac{1}{2} \tau \left(\frac{\partial u}{\partial x} \right)^2 dx \right]_0^t = \int_a^b \frac{1}{2} \tau \left(\frac{\partial u}{\partial x} \right)^2 dx.$$

2.4.3 The Physics of Green's 1st Identity

sec2.4.2

Let $S = u$. Then 2.3 becomes

6 Jan p2

$$\begin{aligned}
\langle u, L_0 u \rangle &= - \left| u^*(x) \tau(x) \frac{d}{dx} u(x) \right|_a^b \\
&\quad + \int_a^b dx \left[\left(\frac{d}{dx} u^* \right) \tau(x) \frac{d}{dx} u(x) + u^*(x) V(x) u(x) \right].
\end{aligned} \tag{2.11}$$

For a *closed string* we have

$$\langle u, L_0 u \rangle = \int_a^b dx \left[\tau(x) \left(\frac{du}{dx} \right)^2 + V(x) (u(x))^2 \right] = 2U \tag{2.12}$$

since each quantity is the same at a and b . For an *open string* we found (equation 1.15)

eq2x

8 Jan p3.2

$$\frac{du}{dx} \Big|_{x=a} = K_a u \tag{2.13}$$

and

$$\frac{du}{dx} \Big|_{x=b} = -K_b u \tag{2.14}$$

so that

$$\begin{aligned}
\langle u, L_0 u \rangle &= \tau(a) K_a |u(a)|^2 + \tau(b) K_b |u(b)|^2 \\
&\quad + \int_a^b dx \left[\tau(x) \left(\frac{du}{dx} \right)^2 + V(x) (u(x))^2 \right] \\
&= 2U,
\end{aligned}$$

twice the potential energy. The term $\frac{1}{2}\tau(a)K_a|u(a)|^2 + \frac{1}{2}\tau(b)K_b|u(b)|^2$ see FW p207
 is the potential energy due to two discrete “springs” at the end points, expl. p10
 and is simply the spring constant times the displacement squared. p126

The term $\tau(x)(du/dx)^2$ is the tension potential energy. Since du/dx eq2y
 represents the string stretching in the transverse direction, $\tau(x)(du/dx)^2$ pr:pe1
 is a potential due to the stretching of the string. $V(x)(u(x))^2$ is the
 elastic potential energy.

For the case of the closed string, equation 2.12, and the open string,
 equation 2.15, the right hand side is equal to twice the potential energy.
 If K_a , K_b , τ and V are positive for the open string, the potential energy
 U is also positive. Thus $\langle u, L_0 u \rangle > 0$, which implies that L_0 is a positive
 definite operator.

pr:pdo1

2.5 Summary

1. The Green's identities are:

- (a) Green's first identity:

$$\begin{aligned} \langle S, L_0 u \rangle = & - \left[S^*(x) \tau(x) \frac{d}{dx} u(x) \right]_a^b \\ & + \int_a^b dx \left[\left(\frac{d}{dx} S^* \right) \tau(x) \frac{d}{dx} u(x) \right. \\ & \left. + S^*(x) V(x) u(x) \right], \end{aligned}$$

- (b) Green's second identity:

$$\langle S, L_0 u \rangle - \langle L_0 S, u \rangle = \left[\tau(x) \left[u(x) \frac{d}{dx} S^*(x) - S^*(x) \frac{d}{dx} u(x) \right] \right]_a^b.$$

2. For a closed string and an open string (i.e., RBC) the linear operator L_0 is Hermitian:

$$\langle S, L_0 u \rangle = \langle u, L_0 S \rangle^*.$$

2.6 References

Green's formula is described in [Stakgold67, p70] and [Stakgold79, p167].

The derivation of the potential energy of a string was inspired by [Simon71,p390].

Chapter 3

Green's Functions

Chapter Goals:

- Show that an external force can be written as a sum of δ -functions.
- Find the Green's function for an open string with no external force on the endpoints.

In this chapter we want to solve the Helmholtz equation, which was obtained in section 1.4.3. First we will develop some mathematical principles which will facilitate the derivation.

8 Jan p3.4
Lagrangian
stuff com-
mented out

3.1 The Principle of Superposition

Suppose that

$$f(x) = a_1 f_1(x) + a_2 f_2(x). \quad (3.1)$$

If u_1 and u_2 are solutions to the equations (c.f., 1.26)

$$[L_0 - \omega^2 \sigma(x)] u_1(x) = \sigma(x) f_1(x) \quad (3.2)$$

$$[L_0 - \omega^2 \sigma(x)] u_2(x) = \sigma(x) f_2(x) \quad (3.3)$$

with RBC and such that (see equation 1.15)

pr:a1.1

eq3q

$$\left. \begin{aligned} (\hat{n}_S \cdot \nabla + \kappa_S) u_1 &= 0 \\ (\hat{n}_S \cdot \nabla + \kappa_S) u_2 &= 0 \end{aligned} \right\} \quad \text{for } x \text{ on } S$$

then their weighted sum satisfies the same equation of motion

$$\begin{aligned} & [L_0 - \omega^2 \sigma(x)](a_1 u_1(x) + a_2 u_2(x)) \\ &= a_1 \underbrace{[L_0 - \omega^2 \sigma(x)]u_1(x)}_{\sigma(x)f_1(x)} + a_2 \underbrace{[L_0 - \omega^2 \sigma(x)]u_2(x)}_{\sigma(x)f_2(x)} \\ &= \sigma(x)f(x). \end{aligned}$$

and boundary condition

$$\begin{aligned} & [\hat{n}_S \cdot \nabla + \kappa_S][a_1 u_1(x) + a_2 u_2(x)] \\ &= a_1 [\hat{n}_S \cdot \nabla + \kappa_S]u_1 + a_2 [\hat{n}_S \cdot \nabla + \kappa_S]u_2 \\ &= a_1(0) + a_2(0) = 0. \end{aligned}$$

We have thus shown that

$$L_0[a_1 u_1 + a_2 u_2] = a_1 L_0 u_1 + a_2 L_0 u_2. \quad (3.4)$$

This is called the *principle of superposition*, and it is the defining property of a *linear operator*.

3.2 The Dirac Delta Function

We now develop a tool to solve the Helmholtz equation (which is also called the steady state equation), equation 1.26:

$$[L_0 - \omega^2 \sigma(x)]u(x) = \sigma(x)f(x).$$

The *delta function* is defined by the equation

$$F_{cd} = \int_c^d dx \delta(x - x_k) = \begin{cases} 1 & \text{if } c < x_k < d \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

where F_{cd} represents the total force over the interval $[c, d]$. Thus we see that the appearance of the delta function is equivalent to the application of a unit force at x_k . The Dirac delta function has units of force/length. On the right-hand side of equation 1.26 make the substitution

$$\sigma(x)f(x) = \delta(x - x_k). \quad (3.6)$$

Integration gives us

$$\int_c^d \sigma(x)f(x)dx = F_{cd}, \quad (3.7)$$

which is the total force applied over the domain. This allows us to write

$$[L_0 - \sigma\omega^2]u(x, \omega) = \delta(x - x_k) \quad a < x < b, \text{RBC} \quad (3.8)$$

where we have written RBC to indicate that the solution of this equation must also satisfy regular boundary conditions. We may now use the principle of superposition to get an arbitrary force. We define an element of such an arbitrary force as

$$F_k = \int_{x_k}^{x_k+\Delta x} dx \sigma(x)f(x) \quad (3.9)$$

$$= \text{the force on the interval } \Delta x. \quad (3.10)$$

We now prove that

$$\sigma(x)f(x) = \sum_{k=1}^N F_k \delta(x - x'_k) \quad (3.11)$$

where $x_k < x'_k < x_k + \Delta x$. We first integrate both sides to get

$$\int_c^d dx \sigma(x)f(x) = \int_c^d dx \sum_{k=1}^N F_k \delta(x - x'_k). \quad (3.12)$$

By definition (equation 3.7), the left-hand side is the total force applied over the domain, F_{cd} . The right-hand side is

$$\int_c^d \sum_{k=1}^N F_k \delta(x - x'_k) dx = \sum_{k=1}^N \int_c^d dx F_k \delta(x - x'_k) \quad (3.13)$$

$$= \sum_{c < x_k < d} F_k \quad (3.14)$$

$$= \sum_{c < x_k < d} \int_{x_k}^{x_k+\Delta x} dx \sigma(x)f(x) \quad (3.15)$$

$$\xrightarrow{N \rightarrow \infty} \int_c^d dx \sigma(x)f(x) \quad (3.16)$$

$$= F_{cd}. \quad (3.17)$$

In the first equality, 3.13, switching the sum and integration holds for all well behaved F_k . Equality 3.14 follows from the definition of the delta function in equation 3.5. Equality 3.15 follows from equation 3.9. By taking the continuum limit, equality 3.16 completes the proof.

pr:Helm2

The Helmholtz equation 3.2 can now be rewritten (using 3.11) as

$$[L_0 - \sigma(x)\omega^2]u(x, \omega) = \sum_{k=1}^N F_k \delta(x - x_k). \quad (3.18)$$

By the principle of superposition we can write

$$u(x) = \sum_{k=1}^N F_k u_k(x) \quad (3.19)$$

where $u_k(x)$ is the solution of $[L_0 - \sigma(x)\omega^2]u_k(x, \omega^2) = \delta(x - x_k)$. Thus, if we know the response of the system to a localized force, we can find the response of the system to a general force as the sum of responses to localized forces.

11 Jan p3

We now introduce the following notation

$$u_k(x) \equiv G(x, x_k; \omega^2) \quad (3.20)$$

pr:Gxxo1

where G is the *Green's function*, x_k signifies the location of the disturbance, and ω corresponds to frequency. This allows us to write

$$\begin{aligned} u(x) &= \sum_{k=1}^N F_k u_k(x) \\ &= \sum_{k=1}^N \int_{x_k}^{x_k + \Delta x} dx' \sigma(x') f(x') G(x, x_k; \omega^2) \\ &\xrightarrow{N \rightarrow \infty} \int_a^b dx' G(x, x'; \omega^2) \sigma(x') f(x'). \end{aligned}$$

We have defined the Green's function by

$$[L_0 - \sigma(x)\omega^2]G(x, x'; \omega^2) = \delta(x - x') \quad a < x, x' < b, \text{RBC.} \quad (3.21)$$

11 Jan p4

pr:NatFreq1

The solution will explode for ω^2 when ω is a natural frequency of the system, as will be seen later.

where will nat
freq be defined

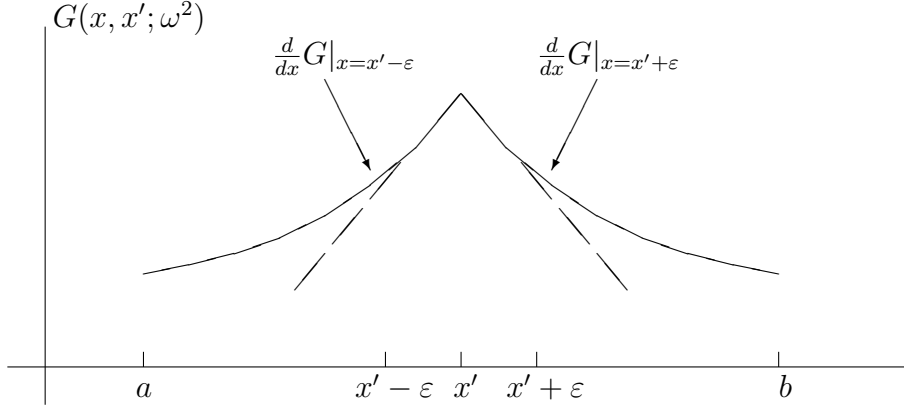


Figure 3.1: The pointed string

Let $\lambda = \omega^2$ be an arbitrary complex number. Since the squared frequency ω^2 cannot be complex, we relabel it λ . So now we want to solve

$$\left[-\frac{d}{dx} \left(\tau(x) \frac{d}{dx} \right) + V(x) - \sigma(x)\lambda \right] G(x, x'; \omega^2) = \delta(x - x') \quad (3.22)$$

$$a < x, x' < b, \text{RBC}$$

Note that G will have singularities when λ is a natural frequency. To obtain a condition which connects solutions on either side of the singularity, we integrate equation 3.22. Consider figure 3.1. In this case

$$\begin{aligned} \int_{x'-\epsilon}^{x'+\epsilon} dx \left[-\frac{d}{dx} \left(\tau(x) \frac{d}{dx} \right) + V(x) - \sigma(x)\lambda \right] G(x, x'; \lambda) \\ = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x') dx \end{aligned}$$

which becomes

$$-\tau(x) \frac{d}{dx} G(x, x'; \lambda) \Big|_{x'-\epsilon}^{x'+\epsilon} = 1 \quad (3.23)$$

since the integrals over $V(x)$ and $\sigma(x)$ vanish as $\epsilon \rightarrow 0$. Note that in this last expression “1” has units of force.

3.3 Two Conditions

11 Jan p5

3.3.1 Condition 1

The previous equation can be written as

$$\frac{d}{dx}G \Big|_{x=x'+\epsilon} - \frac{d}{dx}G \Big|_{x=x'-\epsilon} = -\frac{1}{\tau}. \quad (3.24)$$

eq3other
eq3b

This makes sense after considering that a larger tension implies a smaller kink (discontinuity of first derivative) in the string.

3.3.2 Condition 2

We also require that the string doesn't break:

$$G(x, x')|_{x=x'+\epsilon} = G(x, x')|_{x=x'-\epsilon}. \quad (3.25)$$

eq3d
pr:ContCond1

This is called the *continuity condition*.

3.3.3 Application

pr:homog1

To find the Green's function for equation 3.22 away from the point x' , we study the homogeneous equation

$$[L_0 - \sigma(x)\lambda]u(x, \lambda) = 0 \quad x \neq x', \text{RBC}. \quad (3.26)$$

pr:efp1

This is called the eigen function problem. Once we specify $G(x_0, x'; \lambda)$ and $\frac{d}{dx}G(x_0, x'; \lambda)$, we may use this equation to get all higher derivatives and thus determine $G(x, x'; \lambda)$.

We know, from differential equation theory, that two fundamental solutions must exist. Let u_1 and u_2 be the solutions to

$$[L_0 - \sigma(x)\lambda]u_{1,2}(x, \lambda) = 0 \quad (3.27)$$

pr:AABB

where $u_{1,2}$ denotes either solution. Thus

$$G(x, x'; \lambda) = A_1 u_1(x, \lambda) + A_2 u_2(x, \lambda) \quad \text{for } x < x', \quad (3.28)$$

eq3ab1

and

$$G(x, x'; \lambda) = B_1 u_1(x, \lambda) + B_2 u_2(x, \lambda) \quad \text{for } x > x'. \quad (3.29)$$

We have now defined the Green's function in terms of four constants. We have two matching conditions and two R.B.C.s which determine these four constants.

3.4 Open String

We will solve for an open string with no external force $h(x)$, which was first discussed in section 1.3.2. $G(x, x'; \lambda)$ must satisfy the boundary condition 1.18. Choose u_1 such that it satisfies the boundary condition for the left end

$$-\left. \frac{\partial u_1}{\partial x} \right|_{x=a} + K_a u_1(a) = 0. \quad (3.30)$$

This determines u_1 up to an arbitrary constant. Choose u_2 such that it satisfies the right end boundary condition

$$\left. \frac{\partial u_2}{\partial x} \right|_{x=b} + K_b u_2(b) = 0. \quad (3.31)$$

We find that in equations 3.28 and 3.29, $A_2 = B_1 = 0$. Thus we have two remaining conditions to satisfy.

We now have

$$G(x, x'; \lambda) = A_1(x') u_1(x, \lambda) \quad \text{for } x < x'. \quad (3.32)$$

Note that only the boundary condition at a applies since the behavior of $u_1(x)$ does not matter at b (since $b > x'$). This gives G determined up to an arbitrary constant. We can also write

$$G(x, x'; \lambda) = B_2(x') u_2(x, \lambda) \quad \text{for } x > x'. \quad (3.33)$$

We also note that A and B are constants determined by x' only. Thus we can write the previous expressions in a more symmetric form:

$$G(x, x'; \lambda) = C u_1(x, \lambda) u_2(x', \lambda) \quad \text{for } x < x', \quad (3.34)$$

$$G(x, x'; \lambda) = D u_1(x', \lambda) u_2(x, \lambda) \quad \text{for } x > x'. \quad (3.35)$$

In one of the problem sets we prove that $G(x, x'; \lambda) = G(x', x; \lambda)$. This can also be stated as *Green's Reciprocity Principle*: 'The amplitude of the string at x subject to a localized force applied at x' is equivalent to the amplitude of the string at x' subject to a localized force applied at x .'

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We now apply the continuity condition. Equation 3.25 implies that $C = D$.

Now we have a function symmetric in x and x' , which verifies the Green's Reciprocity Principle. By imposing the condition in equation 3.24 we will be able to determine C :

eq3trione

$$\left. \frac{dG}{dx} \right|_{x=x'+\epsilon} = C \left. \frac{du_1}{dx} \right|_{x'} u_2(x') \quad (3.36)$$

eq3tritwo

$$\left. \frac{dG}{dx} \right|_{x=x'-\epsilon} = C u_1(x') \left. \frac{du_2}{dx} \right|_{x'} \quad (3.37)$$

Combining equations (3.24), (3.36), and (3.37) gives us

$$C \left[u_1 \frac{du_2}{dx} - \frac{du_1}{dx} u_2 \right] \Big|_{x=x'} = \frac{-1}{\tau(x')}. \quad (3.38)$$

pr:wronsk1

The Wronskian is defined as

$$W(u_1, u_2) \equiv u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}. \quad (3.39)$$

This allows us to write

$$C = \frac{1}{-\tau(x') W(u_1(x', \lambda), u_2(x', \lambda))}. \quad (3.40)$$

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Thus

$$G(x, x'; \lambda) = \frac{u_1(x_<, \lambda) u_2(x_>, \lambda)}{-\tau(x') W(u_1(x', \lambda), u_2(x', \lambda))}, \quad (3.41)$$

eq3.39

where we define

pr:xless1

$$u(x_<) \equiv \begin{cases} u(x) & \text{if } x < x' \\ u(x') & \text{if } x' < x \end{cases}$$

and

$$u(x_>) \equiv \begin{cases} u(x) & \text{if } x > x' \\ u(x') & \text{if } x' > x. \end{cases}$$

The u 's are two different solutions to the differential equation:

$$[L_0 - \sigma(x)\lambda]u_1 = 0 \quad [L_0 - \sigma(x)\lambda]u_2 = 0. \quad (3.42)$$

Multiply the first equation by u_2 and the second by u_1 . Subtract one equation from the other to get $-u_2(\tau u_1')' + u_1(\tau u_2')' = 0$ (where we have used equation 1.10, $L_0 = -\frac{\partial}{\partial x}(\tau \frac{\partial}{\partial x}) + V$). Rewriting this as a total derivative gives

$$\frac{d}{dx}[\tau(x)W(u_1, u_2)] = 0. \quad (3.43)$$

This implies that the expression $\tau(x)W(u_1(x, \lambda), u_2(x, \lambda))$ is independent of x . Thus G is symmetric in x and x' .

The case in which the Wronskian is zero implies that $u_1 = \alpha u_2$, since then $0 = u_1 u_2' - u_2 u_1'$, or $u_2'/u_2 = u_1'/u_1$, which is only valid for all x if u_1 is proportional to u_2 . Thus if u_1 and u_2 are linearly independent, the Wronskian is non-zero.

3.5 The Forced Oscillation Problem

The general forced harmonic oscillation problem can be expanded into equations having forces internally and on the boundary which are simple time harmonic functions. Consider the effect of a harmonic forcing term

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(x, t) = \sigma(x)f(x)e^{-i\omega t}. \quad (3.44)$$

We apply the following boundary conditions:

$$\frac{-\partial u(x, t)}{\partial x} + \kappa_a u(x, t) = h_a e^{-i\omega t} \quad \text{for } x = a, \quad (3.45)$$

and

$$\frac{\partial u(x, t)}{\partial x} + \kappa_b u(x, t) = h_b e^{-i\omega t}. \quad \text{for } x = b. \quad (3.46)$$

We want to find the steady state solution. First, we assume a steady state solution form, the time dependence of the solution being

$$u(x, t) = e^{-i\omega t}u(x). \quad (3.47)$$

After making the substitution we get an ordinary differential equation in x . Next determine $G(x, x'; \lambda = \omega^2)$ to obtain the general steady state solution. In the second problem set we use Green's Second Identity to solve this inhomogeneous boundary value problem. All the physics of the exciting system is given by the Green's function.

3.6 Free Oscillation

Another kind of problem is the free oscillation problem. In this case $f(x, t) = 0$ and $h_a = h_b = 0$. The object of this problem is to find the natural frequencies and normal modes. This problem is characterized by the equation:

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(x, t) = 0 \quad (3.48)$$

with the *Regular Boundary Conditions*:

- u is periodic. (*Closed string*)
- $[\hat{n} \cdot \nabla + K_S]u = 0$ for x in S . (*Open string*)

The goal is to find normal mode solutions $u(x, t) = e^{-i\omega_n t} u_n(x)$. The natural frequencies are the ω_n and the natural modes are the $u_n(x)$.

We want to solve the eigenvalue equation

$$[L_0 - \sigma \omega_n^2] u_n(x) = 0 \quad \text{with R.B.C.} \quad (3.49)$$

The variable ω_n^2 is called the eigenvalue of L_0 . The variable $u_n(x)$ is called the eigenvector (or eigenfunction) of the operator L_0 .

3.7 Summary

1. The Principle of superposition is

$$L_0[a_1 u_1 + a_2 u_2] = a_1 L_0 u_1 + a_2 L_0 u_2,$$

where L_0 is a linear operator, u_1 and u_2 are functions, and a_1 and a_2 are constants.

2. The Dirac Delta Function is defined as

$$\int_c^d dx \delta(x - x_k) = \begin{cases} 1 & \text{if } c < x_k < d \\ 0 & \text{otherwise.} \end{cases}$$

3. Force contributions can be constructed by superposition.

$$\sigma(x)f(x) = \sum_{k=1}^N F_k \delta(x - x'_k).$$

4. The Green's Function is the solution to to an equation whose inhomogeneous term is a δ -function. For the Helmholtz equation, the Green's function satisfies:

$$[L_0 - \sigma(x)\omega^2]G(x, x'; \omega^2) = \delta(x - x') \quad a < x, x' < b, \text{RBC.}$$

5. At the source point x' , the Green's function satisfies $\frac{d}{dx}G|_{x=x'+\epsilon} - \frac{d}{dx}G|_{x=x'-\epsilon} = -\frac{1}{\tau}$ and $G(x, x')|_{x=x'+\epsilon} = G(x, x')|_{x=x'-\epsilon}$.
6. Green's Reciprocity Principle is 'The amplitude of the string at x subject to a localized force applied at x' is equivalent to the amplitude of the string at x' subject to a localized force applied at x .'
7. The Green's function for the 1-dimensional wave equation is given by

$$G(x, x'; \lambda) = \frac{u_1(x_<, \lambda)u_2(x_>, \lambda)}{-\tau(x')W(u_1(x', \lambda), u_2(x', \lambda))}.$$

8. The forced oscillation problem is

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(x, t) = \sigma(x)f(x)e^{-i\omega t},$$

with periodic boundary conditions or the elastic boundary conditions with harmonic forcing.

9. The free oscillation problem is

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(x, t) = 0.$$

3.8 Reference

See [Fetter81, p249] for the derivation at the end of section 3.4.

A more complete understanding of the delta function requires knowledge of the theory of distributions, which is described in [Stakgold67a, p28ff] and [Stakgold79, p86ff].

The Green's function for a string is derived in [Stakgold67a, p64ff].

Chapter 4

Properties of Eigen States

13 Jan p7

Chapter Goals:

- Show that for the Helmholtz equation, $\omega_n^2 > 0$, ω_n^2 is real, and the eigen functions are orthogonal.
- Derive the dispersion relation for a closed massless string with discrete mass points.
- Show that the Green's function obeys Hermitian analyticity.
- Derive the form of the Green's function for λ near an eigen value λ_n .
- Derive the Green's function for the fixed string problem.

By definition 2.1

$\omega^2 > 0$

$$\langle S, u \rangle = \int_a^b dx S^*(x) u(x). \quad (4.1)$$

In section 2.4 we saw (using Green's first identity) for L_0 as defined in equation 2.2, and for all u which satisfy equation 1.26, that $V > 0$ implies $\langle u, L_0 u \rangle > 0$. We choose $u = u_n$ and use equation 3.49 so that

$$0 < \langle u_n, L_0 u_n \rangle = \langle u_n, \sigma u_n \rangle \omega_n^2. \quad (4.2)$$

Remember that σ signifies the mass density, and thus $\sigma > 0$. So we conclude

$$\omega_n^2 = \frac{\langle u_n, L_0 u_n \rangle}{\langle u_n, \sigma u_n \rangle} > 0. \quad (4.3)$$

13 Jan p8
 ω^2 real

This all came from Green's first identity.

Next we apply Green's second identity 2.5,

$$\langle S, L_0 u \rangle = \langle L_0 S, u \rangle \quad \text{for } S, u \text{ satisfying RBC} \quad (4.4)$$

Let $S = u = u_n$. This gives us

$$\omega_n^2 \langle u_n, \sigma u_n \rangle = \langle u_n, L_0 u_n \rangle \quad (4.5)$$

$$= \langle L_0 u_n, u_n \rangle \quad (4.6)$$

$$= (\omega_n^2)^* \langle u_n, \sigma u_n \rangle. \quad (4.7)$$

We used equation 3.49 in the first equality, 2.5 in the second equality, and both in the third equality. From this we can conclude that ω_n^2 is real.

orthogonality

Now let us choose $u = u_n$ and $S = u_m$. This gives us

$$\langle u_m, L_0 u_n \rangle = \langle L_0 u_m, u_n \rangle. \quad (4.8)$$

Extracting ω_n^2 gives (note that $\sigma(x)$ is real)

$$\omega_n^2 \langle u_m, \sigma u_n \rangle = \omega_m^2 \langle \sigma u_m, u_n \rangle = \omega_m^2 \langle u_m, \sigma u_n \rangle. \quad (4.9)$$

So

$$(\omega_n^2 - \omega_m^2) \langle u_m, \sigma u_n \rangle = 0. \quad (4.10)$$

Thus if $\omega_n^2 \neq \omega_m^2$ then $\langle u_m, \sigma u_n \rangle = 0$:

$$\int_a^b dx u_m^*(x) \sigma(x) u_n(x) = 0 \quad \text{if } \omega_n^2 \neq \omega_m^2. \quad (4.11)$$

eq4.11

13 Jan p8
pr:ortho1
pr:orthon1

That is, two eigen vectors u_m and u_n of L_0 corresponding to different eigenvalues are orthogonal with respect to the weight function σ . If the eigen vectors u_m and u_n are normalized, then the *orthonormality* condition is

$$\int_a^b dx u_m^*(x) \sigma(x) u_n(x) = \delta_{mn} \quad \text{if } \omega_n^2 \neq \omega_m^2, \quad (4.12)$$

eq4.11p

where the Kronecker delta function is 1 if $m = n$ and 0 otherwise.

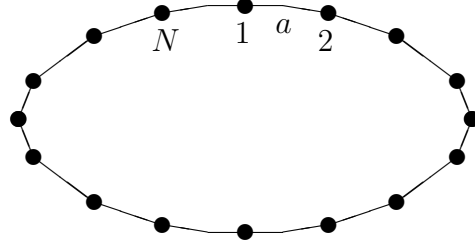


Figure 4.1: The closed string with discrete mass points.

4.1 Eigen Functions and Natural Modes

We now examine the natural mode problem given by equation 3.49. To find the natural modes we must know the natural frequencies ω_n and the normal modes u_n . This is equivalent to the problem

$$L_0 u_n(x) = \sigma(x) \lambda_n u_n(x), \quad \text{RBC.} \quad (4.13)$$

To illustrate this problem we look at a discrete problem.

4.1.1 A Closed String Problem

This problem is illustrated in figure 4.1¹. In this problem the mass density σ and the tension τ are constant, and the potential V is zero. The term $u(x_i)$ represents the perpendicular displacement of the i th mass point. The string density is given by $\sigma = m/a$ where m is the mass of each mass point and a is a unit of length. We also make the definition $c = \sqrt{\tau/\sigma}$. Under these conditions equation 1.1 becomes

$$m\ddot{u}_i = F_{\text{tot}} = \frac{\tau}{a}(u_{i+1} + u_{i-1} - 2u_i).$$

Substituting the solution form $u_i = e^{ikx_i} e^{i\omega t}$ into this equation gives

$$m\omega^2 = 2\frac{\tau}{a} \left(-\frac{e^{ika} + e^{-ika}}{2} + 1 \right) = 2\frac{\tau}{a}(1 - \cos ka) = 4\frac{\tau}{a} \sin^2 \frac{ka}{2}.$$

But continuity implies $u(x) = u(x + Na)$, so $e^{ikNa} = 1$, or $kNa = 2\pi n$, so that

$$k = \left(\frac{2\pi}{Na} \right) n, \quad n = 1, \dots, N.$$

¹See FW p115.

The natural frequencies for this system are then

$$\omega_n^2 = \frac{c^2 \sin^2(k_n a/2)}{a^2/4} \quad (4.14)$$

where $k_n = \frac{2\pi}{L}n$ and n can take on the values $0, \pm 1, \dots, \pm \frac{N-1}{2}$ for odd N , and $0, \pm 1, \dots, \pm \frac{N}{2} - 1, +\frac{N}{2}$ for even N . The constant is $c^2 = a\tau/m$. Equation 4.14 is called the *dispersion relation*. If n is too large, the u_n take on duplicate values. The physical reason that we are restricted to a finite number of natural modes is because we cannot have a wavelength $\lambda < a$. The corresponding normal modes are given by

$$\phi_n(x_i, t) = e^{-i\omega_n t} u_n(x_i) \quad (4.15)$$

$$= e^{-i[\omega_n t - k_n x_i]} \quad (4.16)$$

$$= e^{-i[\omega_n t - \frac{2\pi}{L} n x_i]}. \quad (4.17)$$

The normal modes correspond to traveling waves. Note that ω_n is doubly degenerate in equation 4.14. Solutions $\phi_n(x)$ for n which are larger than allowed give the same displacement of the mass points, but with some nonphysical wavelength. Thus we are restricted to N modes and a cutoff frequency.

4.1.2 The Continuum Limit

We now let a become increasingly small so that N becomes large for L fixed. This gives us $\Delta k_n = 2\pi/L$ for $L = Na$. In the continuum limit, the number of normal modes becomes infinite. Shorter and shorter wavelengths become physically relevant and there is no cutoff frequency.

Letting a approach zero while L remains fixed gives

$$\omega_n^2 = \frac{c^2 \sin^2 \frac{k_n a}{2}}{\frac{a^2}{4}} \xrightarrow{Na=L, a \rightarrow 0} c^2 k_n^2 \quad (4.18)$$

and so

$$\omega_n = c|k_n| \quad (4.19)$$

$$\Delta\omega_n = c|\Delta k_n| = c \left| \frac{2\pi}{L} \right| \quad (4.20)$$

pr:cutoff1
frequency
expression

eq4r

pr:DispRel1

eq4giver

pr:travel1

$$\omega_n = c \frac{2\pi}{L} n. \quad (4.21)$$

Equation (4.17) gives us the u_n 's for all n .

We have found

characteristics

- For a closed string, the two eigenvectors for every eigenvalue (called *degeneracy*) correspond to the two directions in which a wave can move. The eigenvalues are ω_n^2 . pr:degen1
- The natural frequencies ω_n are always discrete, with a separation distance proportional to $1/L$.
- For open strings there is no degeneracy. This is because the position and slope of the Green's function at the ends is fixed by the open string boundary conditions, whereas the closed string boundary conditions do not determine the Green's function at any particular point.

For the discrete closed case, the ω_n 's are discrete with double degeneracy, giving u_{\pm} . We also find the correspondence $\Delta\omega_n \sim c/L$ where $c \sim \sqrt{\tau/\sigma}$. We also found that there is no degeneracy for the open discrete case. 15 Jan p4

4.1.3 Schrödinger's Equation

pr:Schro1

Consider again equation 4.13

$$L_0 u_n(x) = \sigma(x) \lambda_n u_n(x) \quad \text{RBC} \quad (4.22)$$

where

$$L_0 = -\frac{d}{dx} \tau(x) \frac{d}{dx} + V(x). \quad (4.23)$$

We now consider the case in which $\tau(x) = \hbar^2/2m$ and $\sigma = 1$, both quantities being numerical constants. The linear operator now becomes

$$L_0 = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x). \quad (4.24)$$

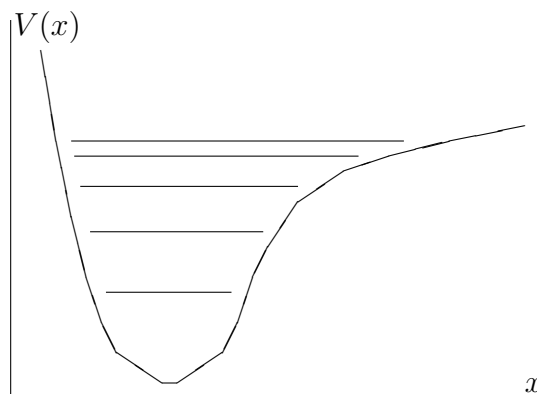


Figure 4.2: Negative energy levels

This is the linear operator for the Schrödinger equation for a particle of mass m in a potential V :

$$\left[\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] u_n(x) = \lambda_n u_n(x) \quad + \text{RBC} \quad (4.25)$$

In this case λ gives the allowed energy values.

The potential $V(x)$ can be either positive or negative. It needs to be positive for L_0 to be positive definite, in which case $\lambda_0 > 0$. For $V < 0$ we can have a finite number of eigenvalues λ less than zero.

verify this
paragraph
15 Jan p5
pr:RayQuo1
fig4neg

On a physical string the condition that $V > 0$ is necessary.

One can prove that negative energy levels are discrete and bounded from below. The bound depends on the nature of $V(x)$ (Rayleigh quotient idea). Suppose that V has a minimum, as shown in figure 4.2 for example. By Green's first identity the quantity $L_0 - V_{\min}$ gives a new operator which is positive definite.

4.2 Natural Frequencies and the Green's Function

pr:Fred1

We now look at a Green's function problem in the second problem set. Under consideration is the Fredholm equation

$$[L_0 - \sigma(x)\omega_n^2]u(x) = \sigma(x)f(x), \quad \text{RBC}. \quad (4.26)$$

In problem 2.2 one shows that the solution $u_n(x)$ for this equation only exists if

$$\int dx u_n^*(x) \sigma(x) f(x) = 0. \quad (4.27)$$

This is the condition that the eigenvectors $u_n(x)$ are orthogonal to the function $f(x)$. We apply this to the Green's function. We choose $\lambda = \lambda_n = \omega_n^2$ and evaluate the Green's function at this point. Thus for the equation

$$[L_0 - \sigma(x)\lambda_n]G(x, x'; \lambda) = \delta(x - x') \quad (4.28)$$

there will be a solution $G(x, x'; \lambda)$ only if (using 4.27)

$$G(x, x; \lambda) = \int dx' G(x, x'; \lambda) \delta(x - x') = 0. \quad (4.29)$$

The result $G(x, x; \lambda) = 0$ implies that $u_n^*(x') = 0$ (using equation 3.41). There will be no solution unless x' is a node. In physical terms, this means that a natural frequency can only be excited at a node.

eq4.26

Ask Baker

Isn't this just hitting a stationary point? eq4.26p

clarify this.

eq4.26b

15 Jan p6

4.3 GF behavior near $\lambda = \lambda_n$

15 Jan p6

From the result of the previous section, we expect that if the driving frequency ω is not a natural frequency, everything will be well behaved. So we show that $G(x, x'; \lambda)$ is good everywhere (in the finite interval $[a, b]$) except for a finite number of points. The value of $G(x, x'; \lambda)$ becomes infinite near λ_n , that is, as $\lambda \rightarrow \lambda_n$. For λ near λ_n we can write the Green's function as

pr:gnxx1

$$G(x, x'; \lambda) \sim \frac{1}{\lambda_n - \lambda} g_n(x, x') + \text{finite} \quad \lambda \rightarrow \lambda_n \quad (4.30)$$

where finite is a value always of finite magnitude. We want to find g_n , so we put $[L_0 - \lambda\sigma(x)]$ in front of each side of the equation and then add and subtract $\lambda_n\sigma(x)G(x, x'; \lambda)$ on the the right-hand side. This gives us (using 4.28)

$$\begin{aligned} \delta(x - x') &= [L_0 - \lambda_n\sigma(x)] \left[\frac{1}{\lambda_n - \lambda} g_n(x, x') + \text{finite} \right] \\ &\quad + (\lambda_n - \lambda)\sigma(x) \left[\frac{1}{\lambda_n - \lambda} g_n(x, x') + \text{finite} \right] \quad \lambda \rightarrow \lambda_n \\ &= [L_0 - \lambda_n\sigma(x)] \left[\frac{1}{\lambda_n - \lambda} g_n(x, x') \right] + \text{finite} \quad \lambda \rightarrow \lambda_n. \end{aligned}$$

15 Jan p7

The left-hand side is also finite if we exclude $x = x'$. This can only occur if

$$[L_0 - \lambda_n \sigma(x)]g_n(x, x') = 0 \quad x \neq x'. \quad (4.31)$$

From this we can conclude that g_n has the form

$$g_n(x, x') = u_n(x)f(x'), \quad x \neq x', \quad (4.32)$$

eq4guf

pr:fxprime1

pr:NatFreq3

where $f(x')$ is a finite term and $u_n(x)$ satisfies $[L_0 - \lambda_n \sigma(x)]u_n(x) = 0$ with RBC. Note that here the eigen functions $u_n(x)$ are not yet normalized. This is the relation between the natural frequency and the Green's function.

4.4 Relation between GF & Eig. Fn.

20 Jan p2

pr:SpecThy1

We continue developing the relation between the Green's function and spectral theory. So far we have discussed the one dimensional problem. This problem was formulated as

$$[L_0 - \sigma(x)\lambda]G(x, x'; \lambda) = \delta(x - x'), \quad \text{RBC}. \quad (4.33)$$

eq4.31m

To solve this problem we first solved the corresponding homogeneous problem

$$[L_0 - \sigma(x)\lambda_n]u_n(x) = 0, \quad \text{RBC}. \quad (4.34)$$

The eigenvalue λ_n is called degenerate if there is more than one u_n per λ_n . We note the following properties in the Green's function:

1. $G^*(x, x'; \lambda^*) = G(x, x'; \lambda)$. Recall that σ , L_0 , and the boundary condition terms are real. First we take the complex conjugate of equation 4.33,

$$[L_0 - \sigma(x)\lambda^*]G^*(x, x'; \lambda) = \delta(x - x'), \quad \text{RBC} \quad (4.35)$$

and then we take the complex conjugate of λ to get

$$[L_0 - \sigma(x)\lambda]G^*(x, x'; \lambda^*) = \delta(x - x'), \quad \text{RBC} \quad (4.36)$$

which gives us

$$G^*(x, x'; \lambda^*) = G(x, x'; \lambda). \quad (4.37)$$

2. G is symmetric. In the second problem set it was seen that

$$G(x, x'; \lambda) = G(x', x; \lambda). \quad (4.38)$$

3. The Green's function G has the property of Hermitian analyticity. 18 Jan p3
By combining the results of 1 and 2 we get pr:HermAn1

$$G^*(x, x'; \lambda) = G(x', x; \lambda^*). \quad (4.39)$$

This may be called the property of *Hermitian analyticity*. eq4ha

In the last section we saw that

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{g_n(x, x')}{\lambda_n - \lambda} \quad (4.40)$$

for g such that

$$[L_0 - \sigma(x)\lambda_n]g_n(x, x') = 0. \quad (4.41)$$

For the open string there is no degeneracy and for the closed string there is double degeneracy. (There is also degeneracy for the 2- and 3-dimensional cases.) give ref

4.4.1 Case 1: λ Nondegenerate

Assume that λ_n is non-degenerate. In this case we can write (using equation 4.32)

$$g_n(x, x') = u_n(x)f_n(x'). \quad (4.42)$$

Hermitian analyticity and the complex conjugate of equation 4.40 give eq4cB
(note that $\lambda_n \in \mathbf{R}$)

$$\frac{g_n^*(x, x')}{\lambda_n - \lambda^*} = \frac{g_n(x', x)}{\lambda_n - \lambda^*} \quad (4.43)$$

as $\lambda \rightarrow \lambda_n$. This implies that

$$g_n^*(x, x') = g_n(x', x). \quad (4.44)$$

So now 4.42 becomes

eq4cC
20 Jan p4

$$u_n^*(x)f_n^*(x') = f_n(x)u_n(x') \quad (4.45)$$

so that (since x and x' are independent) if $u_n(x)$ is normalized (according to equation 4.12)

$$f_n(x) = u_n^*(x). \quad (4.46)$$

In the non-degenerate case we have (from 4.40 and 4.42)

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda}. \quad (4.47)$$

eq4.47

where $u_n(x)$ are normalized eigen functions.

pr:normal1

4.4.2 Case 2: λ_n Double Degenerate

In the second case, the eigenvalue λ_n has double degeneracy like the closed string. The homogeneous closed string equation is

$$(L_0 + \lambda_n \sigma)u_n^{(\pm)}(x) = 0.$$

The eigenfunctions corresponding to λ_n are $u_n^+(x)$ and $u_n^-(x)$. By using the same reasoning that lead to equation 4.47 we can write

why?

$$G(x, x'; \lambda) \rightarrow \frac{1}{\lambda_n - \lambda} [u_n^{(+)}(x)u_n^{(+)*}(x) + u_n^{(-)}(x)u_n^{(-)*}(x)] \quad (4.48)$$

for the equation

$$[L_0 - \sigma(x)\lambda_n]u_n^{(\pm)}(x) = 0, \quad \text{RBC.} \quad (4.49)$$

How does this The eigenfunction u_n may be written as $u_n = A_+u^+ + A_-u^-$. Double fit in? degeneracy is the maximum possible degeneracy in one dimension.

20 Jan p5

In the general case of α -fold degeneracy

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{1}{\lambda_n - \lambda} \sum_{\alpha} [u_n^{\alpha}(x)u_n^{\alpha*}(x')] \quad (4.50)$$

where $u_n^{\alpha}(x)$ solves the equation

$$[L_0 - \sigma(x)\lambda_n]u_n^{\alpha}(x) = 0, \quad \text{RBC.} \quad (4.51)$$

The mathematical relation between the Green's function and the eigen functions is the following: The eigenvalues λ_n are the poles of G . The sum of bilinear products $\sum_{\alpha} u_n^{\alpha}(x)u_n^{\alpha*}(x')$ is the residue of the pole $\lambda = \lambda_n$.

pr:poles1

4.5 Solution for a Fixed String

We want to solve equation 3.22. Further, we take $V = 0$, σ and τ constant, and $a = 0, b = L$.

$$\left[-\tau \frac{d^2}{dx^2} - \lambda \sigma \right] G(x, x'; \lambda) = \delta(x - x') \quad \text{for } 0 < x, x' < L \quad (4.52)$$

for the case of a fixed end string. Our boundary conditions are

$$G(x, x'; \lambda) = 0 \quad \text{for } x = a, b.$$

4.5.1 A Non-analytic Solution

We know from equation 3.41 the solution is

20 Jan p6

$$G(x, x'; \lambda) = \frac{u_1(x_<, \lambda) u_2(x_>, \lambda)}{-\tau W(u_1, u_2)}. \quad (4.53)$$

This solution only applies to the one dimensional case. This is because the solution was obtained using the theory of ordinary differential equations. The corresponding homogeneous equations are given by

eq4.54

pr:homog2

$$\left[\frac{d^2}{dx^2} + \frac{\lambda}{c^2} \right] u_{1,2}(x, \lambda) = 0. \quad (4.54)$$

In this equation we have used the definition $1/c^2 \equiv \sigma/\tau$. The variables u_1 and u_2 also satisfy the conditions

eq4lcu

pr:c1

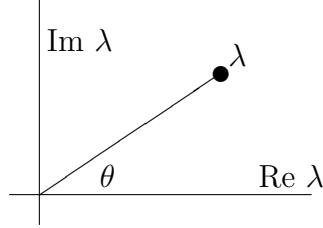
$$u_1(0, \lambda) = 0 \quad \text{and} \quad u_2(L, \lambda) = 0. \quad (4.55)$$

The solution to this homogeneous problem can be found to be

$$u_1 = \sin \sqrt{\frac{\lambda}{c^2}} x \quad \text{and} \quad u_2 = \sin \sqrt{\frac{\lambda}{c^2}} (L - x). \quad (4.56)$$

In these solutions $\sqrt{\lambda}$ appears. Since λ can be complex, we must define a branch cut².

pr:branch1

Figure 4.3: The θ -convention

4.5.2 The Branch Cut

Since $G \sim \frac{1}{\lambda_n - \lambda}$ (see 4.40) it follows (from $\lambda_n > 0$) that G_n is analytic for $\text{Re}(\lambda) < 0$. As a convention, we choose θ such that $0 < \theta < 2\pi$.

fig4one

This is illustrated in figure 4.3.

Using this convention, $\sqrt{\lambda}$ can be represented by

$$\sqrt{\lambda} = \sqrt{|\lambda|} e^{i\theta/2} \quad (4.57)$$

$$= \sqrt{|\lambda|} \left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right]. \quad (4.58)$$

eq4.58-59

20 Jan p7

Note that $\sqrt{\lambda}$ has a discontinuity along the positive real axis. The function G is analytic in the complex plane if the positive real axis is removed. This can be expressed mathematically as the condition that

$$\text{Im} \sqrt{\lambda} > 0. \quad (4.59)$$

4.5.3 Analytic Fundamental Solutions and GF

pr:u1.1

We now look back at the fixed string problem. We found that

$$u_1(x, \lambda) = \sin \sqrt{\frac{\lambda}{c^2}} x$$

so that

$$\frac{du_1}{dx} = \sqrt{\frac{\lambda}{c^2}} \cos \sqrt{\frac{\lambda}{c^2}} x.$$

but only if $x = a = 0$. This gives us the boundary value

²See also FW p485.

$$\left. \frac{du_1}{dx} \right|_{x=a} = \sqrt{\frac{\lambda}{c^2}}. \quad (4.60)$$

Because of the $\sqrt{\lambda}$, this is not analytic over the positive real axis. We choose instead the solution

$$u_1 \rightarrow \frac{u_1}{\left. \frac{du_1}{dx} \right|_{x=a}} = \frac{1}{\sqrt{\frac{\lambda}{c^2}}} \sin \sqrt{\frac{\lambda}{c^2}} x \equiv \bar{u}_1. \quad (4.61)$$

The function \bar{u}_1 has the properties

$$\bar{u}_1(a) = 0 \quad \text{and} \quad \left. \frac{d\bar{u}_1}{dx} \right|_{x=a} = 1.$$

This satisfies the differential equation.

18 Jan p8

$$\left[\frac{d^2}{dx^2} + \frac{\lambda}{c^2} \right] \bar{u}_1(x, \lambda) = 0. \quad (4.62)$$

So \bar{u}_1 is analytic for λ with no branch cut. Similarly, for the substitution $\bar{u}_2 \equiv u_2 / (\left. \frac{du_2}{dx} \right|_{x=b})$ we obtain justify this

$$\bar{u}_2 = (\lambda/c^2)^{-1/2} \sin \sqrt{\lambda/c^2} (L - x).$$

One can always find u_1 and u_2 as analytic functions of λ with no branch cut.

Is this always true?

4.5.4 Analytic GF for Fixed String

We have been considering the Green's function equation

$$[L_0 - \lambda \sigma(x)] G(x, x'; \lambda) = \delta(x - x') \quad \text{for } 0 < x, x' < b \quad (4.63)$$

with the open string RBC (1.18)

$$[\hat{n}_S \cdot \nabla + \kappa_S] G(x, x'; \lambda) = 0 \quad \text{for } x \text{ on } S \quad (4.64)$$

and linear operator L_0 defined as

$$L_0 = -\frac{d}{dx} \tau(x) \frac{d}{dx} + V(x). \quad (4.65)$$

We found that the solution to this equation can be written (3.41)

$$G(x, x'; \lambda) = \frac{u_1(x_<, \lambda)u_2(x_>, \lambda)}{-\tau(x)W(u_1, u_2)} \quad (4.66)$$

eq4cD

where u_1 and u_2 are solutions to the homogeneous equation with the same boundary conditions as (1.18)

$$[L_0 - \lambda\sigma(x)]u_{1,2}(x, \lambda) = 0 \quad \text{for } a < x < b \quad (4.67)$$

$$-\frac{\partial}{\partial x}u_1(x, \lambda) + k_a u_1(x, \lambda) = 0 \quad \text{for } x = a \quad (4.68)$$

$$+\frac{\partial}{\partial x}u_2(x, \lambda) + k_b u_2(x, \lambda) = 0 \quad \text{for } x = b. \quad (4.69)$$

We have been calculating the Green's function for a string with fixed tension ($\tau = \text{constant}$) and fixed string density ($\sigma = \text{constant}$) in the absence of a potential field ($V = 0$) and fixed end points. This last condition implies that the Green's function is restricted to the boundary condition that $G = 0$ at $x = a = 0$ and $x = b = L$. We saw that

$$u_1 = \sin \sqrt{\frac{\lambda}{c^2}}x \quad (4.70)$$

and

$$u_2 = \sin \sqrt{\frac{\lambda}{c^2}}(L - x). \quad (4.71)$$

22 Jan p2

We also assigned the convention that

$$\sqrt{\lambda} = \sqrt{|\lambda|}e^{i\theta/2}. \quad (4.72)$$

This is shown in figure 4.3.

The Wronskian in equation 4.53 for this problem can be simplified as

$$\begin{aligned} W(u_1, u_2) &= u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_1}{\partial x} \\ &= \sqrt{\frac{\lambda}{c^2}} \left[-\sin \sqrt{\frac{\lambda}{c^2}}x \cos \sqrt{\frac{\lambda}{c^2}}(L - x) \right. \end{aligned}$$

$$\begin{aligned}
& - \sin \sqrt{\frac{\lambda}{c^2}}(L-x) \cos \sqrt{\frac{\lambda}{c^2}}x \Big] \\
& = -\sqrt{\frac{\lambda}{c^2}} \sin \sqrt{\frac{\lambda}{c^2}}L.
\end{aligned} \tag{4.73}$$

Thus we can write the full solution for the fixed string problem as eq4.75b

$$G(x, x'; \lambda) = \frac{\sin \sqrt{\frac{\lambda}{c^2}}x_{<} \sin \sqrt{\frac{\lambda}{c^2}}(L-x_{>})}{\tau \sqrt{\frac{\lambda}{c^2}} \sin \sqrt{\frac{\lambda}{c^2}}L}. \tag{4.74}$$

eq4ss

4.5.5 GF Properties

We may now summarize the properties of G in the complex λ plane.

- *Branch Cut:* G has no branch cut. It is analytic except at isolated simple poles. The $\lambda^{1/2}$ branch vanishes if the eigen functions are properly chosen. This is a general result for discrete spectrum. pr:DiscSpec1
- *Asymptotic limit:* G goes to zero as λ goes to infinity. If λ is real and we do not go through the poles, this result can be seen immediately from equation 4.74. For complex λ , we let $|\lambda| \rightarrow \infty$ and use the definition stated in equation 4.57 which is valid for $0 < \theta < 2\pi$. This definition gives $\text{Im}\sqrt{\lambda} > 0$. We can then write Justify
pr:asympl

$$\sin \sqrt{\lambda/c^2}x = \frac{e^{i\sqrt{\lambda/c^2}x} - e^{-i\sqrt{\lambda/c^2}x}}{2i} \tag{4.75}$$

$$\xrightarrow{|\lambda| \rightarrow \infty} \frac{e^{-i\sqrt{\lambda/c^2}x}}{2i} \tag{4.76}$$

for $\theta > 0$. Thus from equation 3.44 we get

23 Jan p3

$$G(x, x'; \lambda) \xrightarrow{|\lambda| \rightarrow \infty} -\frac{1}{2i} \frac{e^{-i\sqrt{\lambda/c^2}x_{<}} e^{-i\sqrt{\lambda/c^2}(L-x_{>})}}{\tau \sqrt{\lambda/c^2} e^{-i\sqrt{\lambda/c^2}L}} \tag{4.77}$$

$$= -\frac{1}{2i} \frac{e^{+i\sqrt{\lambda/c^2}(x_{>} - x_{<})}}{\tau \sqrt{\lambda/c^2}}. \tag{4.78}$$

By convention $x_{>} - x_{<} > 0$, and thus we conclude

$$G(x, x'; \lambda) \xrightarrow{|\lambda| \rightarrow \infty} 0. \quad (4.79)$$

eq4.81a

- *Poles:* The Green's function can have poles. The Green's function is a ratio of analytic functions. Thus the poles occur at the zeros of the denominator.

We now look at $\sin \sqrt{\frac{\lambda_n}{c^2}} L = 0$ from the denominator of equation 3.44. The poles are at $\lambda = \lambda_n$. We can write $\sqrt{\lambda_n/c^2} L = n\pi$ or

$$\lambda_n = \left(\frac{cn\pi}{L} \right)^2 \quad \text{for } n = 1, 2, \dots \quad (4.80)$$

eq4et

We delete the case $n = 0$ since we have a removable singularity at $\lambda = 0$. Equation 4.80 occurs when the Wronskian vanishes. This happens when $u_1 = \text{constant} \times u_2$ (not linearly independent). Both u_1 and u_2 satisfy the boundary conditions at both boundaries and are therefore eigenfunctions. Thus the u_n 's are eigenfunctions and the λ_n 's are the eigenvalues. So

23 Jan p4

$$[L_0 - \lambda_n \sigma] u_n(x) = 0 \quad \text{RBC} \quad (4.81)$$

eq4star

is satisfied for λ_n by u_n .

4.5.6 The GF Near an Eigenvalue

We now look at equation 4.74 near an eigenvalue. First we expand the denominator in a power series about $\lambda = \lambda_n$:

$$\sin \sqrt{\frac{\lambda}{c^2}} L = \frac{(\lambda - \lambda_n) \frac{L}{c} \cos \sqrt{\lambda_n} \frac{L}{c}}{2\sqrt{\lambda_n}} + O(\lambda - \lambda_n)^2. \quad (4.82)$$

eq4.85

So for λ near an eigenvalue we have

$$\tau \sqrt{\frac{\lambda}{c^2}} \sin \sqrt{\frac{\lambda}{c^2}} L \xrightarrow{\lambda \rightarrow \lambda_n} \frac{\tau L}{2c^2} (\lambda - \lambda_n) \cos n\pi. \quad (4.83)$$

eq4.86

Now we look at the numerator of 4.74. We can rewrite

$$\sin \sqrt{\lambda/c^2}(L - x_>) = -\sin n\pi x_> \cos \sqrt{\lambda/c^2}L. \quad (4.84)$$

eq4.87

Note that $f(x_<)f(x_>) = f(x)f(x')$. So, with $\sigma = \tau/c^2$, and substituting 4.83 and 4.84, equation 4.74 becomes

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{2}{\sigma L} \frac{\sin \frac{n\pi x}{L} \sin \frac{n\pi x'}{L}}{\lambda_n - \lambda}. \quad (4.85)$$

So we conclude that

$$G \xrightarrow{\lambda \rightarrow \lambda_n} \frac{u_n(x)u_n(x')}{\lambda_n - \lambda} \quad (4.86)$$

as in 4.47 where the eigenfunction is

eq4cC2

22 Jan p5

$$u_n(x) = \sqrt{\frac{2}{\sigma L}} \sin \frac{n\pi x}{L}, \quad (4.87)$$

which satisfies the completeness relation $\int_0^L u_m(x)u_n^*(x)\sigma dx = \delta_{mn}$.

eq4.91

pr:CompRel1

4.6 Derivation of GF form near E.Val.

4.6.1 Reconsider the Gen. Self-Adjoint Problem

We now give an indirect proof of equation 4.86 based on the specific Green's function defined in equation 4.53,

$$G(x, x'; \lambda) = \frac{\bar{u}_1(x_<, \lambda)\bar{u}_2(x_>, \lambda)}{-\tau(x)W(\bar{u}_1, \bar{u}_2)}. \quad (4.88)$$

The boundary conditions are (see 1.18)

eq4nt

$$-\frac{\partial \bar{u}_1}{\partial x} + k_a \bar{u}_1 = 0 \quad \text{for } x = a = 0 \quad (4.89)$$

$$+\frac{\partial \bar{u}_2}{\partial x} + k_b \bar{u}_2 = 0 \quad \text{for } x = b = L. \quad (4.90)$$

The function \bar{u}_1 (respectively \bar{u}_2) may be any solution which is an analytic function of λ , and independent of λ at $x = a$ (respectively

$x = b$). Thus both the numerator and the denominator of equation 4.88 are analytic functions of λ , so there is no branch cut. Note that $G(x, x'; \lambda)$ may only have poles when $W(\bar{u}_1(x, \lambda)\bar{u}_2(x, \lambda)) = 0$, which only occurs when

$$\bar{u}_1(x, \lambda_n) = d_n \bar{u}_2(x, \lambda_n). \quad (4.91)$$

where the d_n are constants.

Look at the Green's function near $\lambda = \lambda_n$. Finding the residue will give the correct normalization. We have (using 4.73 and 4.82)

$$\tau(x)W(\bar{u}_1, \bar{u}_2) \xrightarrow{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n)c_n, \quad (4.92)$$

where c_n is some normalization constant. In this limit equation 4.88 becomes

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{\frac{1}{d_n} \bar{u}_n(x_{<}) \bar{u}_n(x_{>})}{(\lambda - \lambda_n)c_n} \quad (4.93)$$

where d_n is some constant. So $u_n \equiv \bar{u}_n \sqrt{\frac{1}{c_n d_n}}$ is the normalized eigenfunction. Equation 4.88 then implies

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{u_n(x)u_n^*(x')}{(\lambda - \lambda_n)} \quad (4.94)$$

where u_n satisfies equation 4.81.

4.6.2 Summary, Interp. & Asymptotics

In the previous sections we looked at the eigenvalue problem

$$[L_0 - \lambda_n \sigma(x)]u_n(x) = 0 \quad \text{for } a < x < b, \text{ RBC} \quad (4.95)$$

and the Green's function problem

$$[L_0 - \lambda_n \sigma(x)]G(x, x'; \lambda) = \delta(x - x') \quad \text{for } a < x, x' < b, \text{ RBC} \quad (4.96)$$

where

$$L_0 = -\frac{d}{dx} \left(\tau(x) \frac{d}{dx} \right) + V(x), \quad (4.97)$$

which is a formally self-adjoint operator. The general problem requires

finding an explicit expression for the Green's function for a force localized at x' .

For $\lambda \rightarrow \lambda_n$ we found

$$G(x, x'; \lambda) \rightarrow \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda}. \quad (4.98)$$

This equation shows the contribution of the n th eigenfunction. We saw *eq4.102* that G is an analytic function of λ (with poles at λ_n) where $G \rightarrow 0$ as $|\lambda| \rightarrow \infty$. We can think of G as the inverse operator of $L_0 - \lambda_n \sigma$:

$$G = \frac{1}{L_0 - \lambda_n \sigma}. \quad (4.99)$$

Thus the poles are at $L_0 = \lambda_n \sigma$.

For large λ the behavior of G is determined by the $\frac{d^2}{dx^2}G$ term since it brings down the highest power of λ . Thus for our simple example of τ constant,

$$L_0 \approx -\tau \frac{d^2}{dx^2} \quad (4.100)$$

and for λ large

$$G \sim \exp(i\sqrt{\lambda/c^2}x), \quad G' \sim \sqrt{\lambda}G, \quad G'' \sim (\sqrt{\lambda})^2G \quad (4.101)$$

where the derivatives are taken with respect to x .

4.7 General Solution form of GF

25 Jan p2

In this section we obtain a general form (equation 4.108) for the Green's function which is constructed using the solutions to the corresponding eigen value equation. This is done by evaluating a particular complex integral. We have seen that $G(x, x'; \lambda)$ is analytic in the complex λ -plane except for poles on the real axis at the eigen values λ_n . *fig4.1ff*

We consider the following complex integral

$$\oint_{c_1+c_2} \frac{d\lambda' G(x, x'; \lambda')}{\lambda' - \lambda} \equiv \oint_{c_1+c_2} d\lambda' F(\lambda') \quad (4.102)$$

where we have defined $F(\lambda') \equiv G(x, x'; \lambda')/(\lambda' - \lambda)$. Let the contour of integration be the contour illustrated in figure 4.4. This equation has

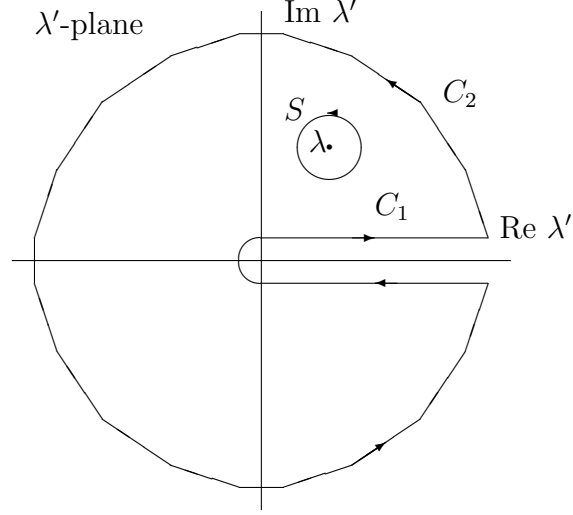


Figure 4.4: The contour of integration

a singularity only at λ . Thus we need only integrate on the contour around λ . This is accomplished by deforming the contour $C_1 + C_2$ to the contour S (following Cauchy's theorem)

$$\oint_{C_1+C_2} d\lambda' F(\lambda') = \oint_S d\lambda' F(\lambda'). \quad (4.103)$$

See figure 4.5. Note that although $F(\lambda')$ blows up as λ' approaches λ , $G(x, x'; \lambda') \rightarrow G(x, x'; \lambda)$ in this limit. Thus the integration about the small circle around λ can be written

$$\oint_S d\lambda' F(\lambda') \xrightarrow{\lambda' \rightarrow \lambda} G(x, x'; \lambda) \oint_S \frac{d\lambda'}{\lambda' - \lambda}.$$

Now make the substitution

$$\lambda' - \lambda = \epsilon e^{i\alpha} \quad (4.104)$$

$$d\lambda' = i\epsilon e^{i\alpha} d\alpha \quad (4.105)$$

$$\frac{d\lambda'}{\lambda' - \lambda} = i d\alpha. \quad (4.106)$$

This allows us to write

$$\oint_S \frac{d\lambda'}{\lambda' - \lambda} = i \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} d\alpha = 2\pi i.$$



Figure 4.5: Circle around a singularity.

We conclude

$$\oint_S d\lambda' F(\lambda') = 2\pi i G(x, x'; \lambda)$$

and thus

$$2\pi i G(x, x'; \lambda) = \oint_{C_1+C_2} d\lambda' \frac{G(x, x'; \lambda')}{\lambda' - \lambda}. \quad (4.107)$$

eq4tpig

We now assume that $G(x, x'; \lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. We must check this for each example we consider. An intuitive reason for this limit is the following. The Green's function is like the inverse of the differential operator: $G \sim 1/(L_0 - \lambda\sigma)$. Thus as λ becomes large, G must vanish. This assumption allows us to evaluate the integral around the large circle C_2 . We parameterize λ' along this contour as

$$\lambda' - \lambda = Re^{i\alpha},$$

$$d\lambda' \rightarrow Re^{i\alpha} i d\alpha \quad \text{as } R \rightarrow \infty.$$

So

$$\lim_{R \rightarrow \infty} \oint_{C_2} \frac{G(x, x'; \lambda')}{\lambda' - \lambda} d\lambda' = \lim_{R \rightarrow \infty} \int_0^{2\pi} \frac{Re^{i\alpha} d\alpha}{Re^{i\alpha}} G(x, x'; Re^{i\theta}) \rightarrow 0$$

since $G(x, x'; Re^{i\theta}) \rightarrow 0$ as $R \rightarrow \infty$.

fig4f of

We now only need to evaluate the integral for the contour C_1 .

$$\oint_{C_1} \frac{d\lambda' G(x, x'; \lambda')}{\lambda' - \lambda} = \sum_n \oint_{c_n} \frac{d\lambda' G(x, x'; \lambda')}{\lambda' - \lambda}.$$

In this equation we replaced the contour C_1 by a sum of contours around the poles, as shown in figure 4.6. Recall that

fig4.6

$$G(x, x'; \lambda') \xrightarrow{\lambda' \rightarrow \lambda_n} \frac{\sum_\alpha u_n^\alpha(x) u_n^{\alpha*}(x')}{\lambda_n - \lambda'}.$$

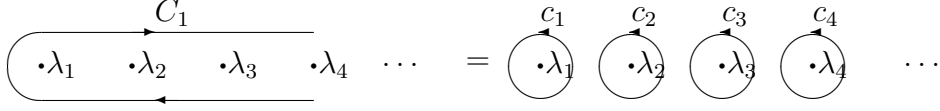


Figure 4.6: Division of contour.

We note that

$$\begin{aligned}
 \oint_{c_n} \frac{d\lambda'}{(\lambda' - \lambda)(\lambda_n - \lambda')} &= \frac{1}{\lambda_n - \lambda} \oint_{c_n} \frac{d\lambda'}{\lambda_n - \lambda'} \\
 &= \frac{1}{\lambda_n - \lambda} \oint \frac{d\lambda'}{\lambda' - \lambda_n} \\
 &= \frac{1}{\lambda_n - \lambda} 2\pi i.
 \end{aligned}$$

The first equality is valid since $1/(\lambda' - \lambda)$ is well behaved as $\lambda' \rightarrow \lambda_n$. The last equality follows from the same change of variables performed above. The integral along the small circle containing λ_n is thus

$$\oint_{c_n} \frac{d\lambda'}{\lambda' - \lambda} G(x, x'; \lambda') = \frac{2\pi i}{\lambda_n - \lambda} \sum_{\alpha} u_n^{\alpha}(x) u_n^{\alpha*}(x').$$

The integral along the contour C_1 (and thus the closed contour $C_1 + C_2$) is then

$$\oint_{C_1} \frac{d\lambda'}{\lambda' - \lambda} G(x, x'; \lambda') = 2\pi i \sum_n \frac{\sum_{\alpha} u_n^{\alpha}(x) u_n^{\alpha*}(x')}{\lambda_n - \lambda}.$$

Substituting equation 4.107 gives the result

$$G(x, x'; \lambda) = \sum_n \frac{1}{\lambda_n - \lambda} \left(\sum_{\alpha} u_{\lambda_n}^{\alpha}(x) u_{\lambda_n}^{\alpha*}(x') \right), \quad (4.108)$$

eq4.124

where the indices λ_n sum $n = 1, 2, \dots$ and α sums over the degeneracy.

4.7.1 δ -fn Representations & Completeness

Using the above result (equation 4.108) we can write

$$\delta(x - x') = [L_0 - \lambda\sigma(x)]G(x, x'; \lambda) \quad (4.109)$$

$$= [L_0 - \lambda\sigma(x)] \sum_{n=1}^{\infty} \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda} \quad (4.110)$$

$$= \sum_{n=1}^{\infty} [(L_0 - \lambda_n\sigma) + (\lambda_n - \lambda)\sigma] \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda} \quad (4.111)$$

$$= \sum_{n=1}^{\infty} \sigma u_n(x)u_n^*(x'). \quad (4.112)$$

In the last equality we used

$$\sum_{n=1}^{\infty} (L_0 - \lambda_n\sigma) \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda} = \sum_{n=1}^{\infty} \frac{u_n(x')}{\lambda_n - \lambda} (L_0 - \lambda_n\sigma)u_n(x) = 0$$

since L_0 is a differential operator in terms of x , From this we get the completeness relation

pr:CompRel2

$$\delta(x - x') = \sum_{n=1}^{\infty} \sigma(x)u_n(x)u_n^*(x') \quad (4.113)$$

or

$$\frac{\delta(x - x')}{\sigma(x)} = \sum_{n=0}^{\infty} u_n(x)u_n^*(x'). \quad (4.114)$$

This is called the completeness relation because it is only true if the u_n are a complete orthonormal set of eigenfunctions, which means that any $f(x)$ can be written as a sum of the u_n 's weighted by the projection of $f(x)$ onto them. This notion is expressed by the expansion theorem (4.117 and 4.118). eq4.128b

We now derive the expansion theorem. Consider

pr:ExpThm1

$$f(x) = \int_a^b dx' f(x') \delta(x - x') \quad \text{for } a < x < b \quad (4.115)$$

25 Jan p6

$$= \int_a^b dx' f(x') \sigma(x') \sum_{n=1}^{\infty} u_n(x)u_n^*(x') \quad (4.116)$$

So

eq4.129

$$f(x) = \sum_{n=1}^{\infty} u_n(x) f_n, \quad (4.117)$$

where

$$f_n = \int_a^b dx' u'(x') \sigma(x') f(x') \quad (4.118)$$

eq4.132

pr:FourCoef1

is the generalized n th Fourier coefficient for $f(x)$. Equation 4.117 represents the projection of $f(x)$ onto the $u_n(x)$ normal modes. This was obtained using the completeness relation.

pr:normal2

Now we check normalization. The Green's function G is normalized because the u 's are normalized. We check the normalization of the u 's by looking at the completeness relation

$$\delta(x - x') = \sigma(x') \sum_n u_n(x) u_n^*(x'). \quad (4.119)$$

Integrate both sides by $\int dx' u_m(x')$. On the left hand side we immediately obtain $\int dx' u_m(x') \delta(x - x') = u_m(x)$. On the right hand side

$$\int dx' u_m(x') \sigma(x') \sum_n u_n(x) u_n^*(x') = \sum_n u_n(x) \int dx' u_n^*(x') \sigma(x') u_m(x')$$

where we used 4.11 in the equality. But

$$u_m(x) = \sum_n u_n(x) \delta_{n,m}.$$

Thus we conclude that normalized eigen functions are used in the completeness relation:

$$\int_a^b dx u_m^*(x) \sigma(x) u_n(x) = \delta_{n,m}. \quad (4.120)$$

pr:orthon2

This is the condition for *orthonormality*.

4.8 Extension to Continuous Eigenvalues

27 Jan p1

27 Jan p2

As L (the length of the string) becomes large, the eigen values become closer together. The normalized eigen functions $u_n(x)$ and eigen values λ_n for the fixed string problem, equation 4.54, can be written

$$u_n = \frac{1}{\sqrt{\sigma L}} e^{\pm i \sqrt{\lambda_n / c^2} x} \quad (4.121)$$

factor of 2?

and

$$\lambda_n = (ck_n)^2 \quad k_n = \frac{2\pi n}{L} \quad n = 0, 1, 2, \dots \quad (4.122)$$

The separation between the eigen values is then

$$\Delta\lambda_n = \lambda_n - \lambda_{n-1} \sim c^2 \left(\frac{2\pi}{L}\right)^2 (n^2 - (n-1)^2) \xrightarrow{L \rightarrow \infty} 0. \quad (4.123)$$

We now consider the case of continuous eigen values. Let λ be complex (as before) and let $\Delta\lambda_n \rightarrow 0$. This limit exists as long as λ is not on the positive real axis, which means that the denominator will not blow up as $\Delta\lambda_n \rightarrow 0$. In the continuum case equation 4.108 becomes

$$\lim_{\Delta\lambda_n \rightarrow 0} G(x, x'; \lambda) = \int \frac{d\lambda_n}{\lambda_n - \lambda} \sum_{\alpha} u_{\lambda_n}^{\alpha}(x) u_{\lambda_n}^{\alpha*}(x). \quad (4.124)$$

eq4.124ab

The completeness relation, equation 4.114, becomes

$$\frac{\delta(x - x')}{\sigma(x')} = \int d\lambda_n \sum_{\alpha} u_{\lambda_n}(x) u_{\lambda_n}^*(x'). \quad (4.125)$$

Now we take any function $f(x)$ and express it as a superposition using the δ -function representation (in direct analogy with equations 4.115 and 4.118 in the discrete case)

$$f(x) = \int d\lambda_n \sum_{\alpha} f_{\lambda_n}^{\alpha} u_{\lambda_n}^{\alpha}(x). \quad (4.126)$$

This is the generalized Fourier integral, with generalized Fourier coefficients eq4.175

$$f_{\lambda_n}^{\alpha} = \int dx u_{\lambda_n}^{\alpha*}(x) \sigma(x) f(x). \quad (4.127)$$

pr:GenFourInt1

The coefficients $f_{\lambda_n}^{\alpha}$ may be interpreted as the projection of $f(x)$ with respect to $\sigma(x)$ onto the eigenfunction $u_{\lambda_n}^{\alpha}(x)$. eq4.176

4.9 Orthogonality for Continuum

27 Jan p3

We now give the derivation of the orthogonality of the eigenfunctions for the continuum case. The method of derivation is the same as we

used in the discrete spectrum case. First we choose $f(x) = u_{\lambda_m}^\alpha(x)$ for $f(x)$ in equation 4.126. Equation 4.127 then becomes

$$f_{\lambda_n'}^{\alpha'*} = \int dx u_{\lambda_n'}^{\alpha'}(x) \sigma u_{\lambda_m}^\alpha(x). \quad (4.128)$$

The form of equation 4.126 corresponding to this is

$$u_{\lambda_m}^\alpha(x) = \sum_\alpha \int d\lambda'_m f_{\lambda_n'}^{\alpha'} u_{\lambda_n'}^{\alpha'}(x). \quad (4.129)$$

This equation can only be true if

$$f_{\lambda_n'}^{\alpha'} = \delta_{\alpha\alpha'} \delta(\lambda'_n - \lambda_m). \quad (4.130)$$

So we conclude that

$$\int dx u_{\lambda_n'}^{\alpha'}(x) \sigma u_{\lambda_m}^\alpha(x) = f_{\lambda_n'}^{\alpha'} = \delta_{\alpha\alpha'} \delta(\lambda'_n - \lambda_m). \quad (4.131)$$

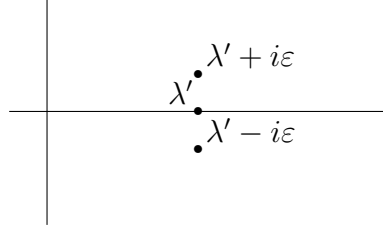
This is the statement of orthogonality, analogous to equation 4.114. All of these results come from manipulations on equation 4.108. We now have both a Fourier sum theorem and a Fourier integral theorem.

We now investigate equation 4.124 in more detail

$$G(x, x'; \lambda) = \int \frac{d\lambda_n}{\lambda_n - \lambda} \left(\sum_\alpha u_{\lambda_n}^\alpha(x) u_{\lambda_n}^\alpha(x') \right).$$

(Generally, the integration is over the interval from zero to infinity.) Where are the singularities? Consider λ approaching the positive real axis. It can't ever get there. The value approaching the negative side may be different from the value approaching the positive side. Therefore this line must be a branch cut corresponding to a continuous spectrum. Note that generally there is only a positive continuous spectrum, although we may have a few negative bound states. So $G(x, x'; \lambda)$ is analytic on the entire complex cut λ -plane. It is in the region of non-analyticity that all the physics occurs. The singular difference of a branch cut is the difference in the value of G above and below. See figure 4.7.

We now examine the branch cut in more detail. Using equation

Figure 4.7: λ near the branch cut.

4.108 we can write

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{G(x, x'; \lambda' + i\epsilon) - G(x, x'; \lambda' - i\epsilon)}{2\pi i} \\ &= \frac{1}{2\pi i} \int_0^\infty d\lambda_n \sum_\alpha u_{\lambda_n}^\alpha(x) u_{\lambda_n}^{\alpha*}(x') \left(\frac{1}{\lambda_n - \lambda' - i\epsilon} - \frac{1}{\lambda_n - \lambda' + i\epsilon} \right) \end{aligned}$$

where

$$\begin{aligned} \frac{1}{2\pi i} \left(\frac{1}{\lambda_n - \lambda' - i\epsilon} - \frac{1}{\lambda_n - \lambda' + i\epsilon} \right) &= \frac{1}{2\pi i} \frac{2i\epsilon}{(\lambda_n - \lambda')^2 + \epsilon^2} \\ &= \frac{\epsilon}{\pi} \frac{1}{(\lambda_n - \lambda')^2 + \epsilon^2}. \end{aligned}$$

In the first problem set we found that

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi} \frac{1}{(\lambda - \lambda')^2 + \epsilon^2} = \delta(\lambda' - \lambda). \quad (4.132)$$

So

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{G(x, x'; \lambda' + i\epsilon) - G(x, x'; \lambda' - i\epsilon)}{2\pi i} \\ &= \int d\lambda_n \sum_\alpha u_{\lambda_n}^\alpha(x) u_{\lambda_n}^{\alpha*}(x') \delta(\lambda' - \lambda_n) \\ &= \sum_\alpha u_{\lambda'}^\alpha(x) u_{\lambda'}^{\alpha*}(x'). \end{aligned}$$

Therefore the discontinuity gives the product of the eigenfunctions.

27 Jan p5

We derived in the second problem set the property

$$G^*(x, x'; \lambda) = G(x, x'; \lambda^*). \quad (4.133)$$

Now take $\lambda = \lambda' + i\epsilon$. This allows us to write

$$G^*(x, x'; \lambda' + i\epsilon) = G(x, x'; \lambda' - i\epsilon) \quad (4.134)$$

and

$$\frac{G(x, x'; \lambda' + i\epsilon) - G(x, x'; \lambda' - i\epsilon)}{2\pi i} \quad (4.135)$$

$$= \frac{G(x, x'; \lambda' + i\epsilon) - G^*(x, x'; \lambda' + i\epsilon)}{2\pi i} \quad (4.136)$$

$$= \frac{1}{\pi} \text{Im} G(x, x'; \lambda' + i\epsilon) \quad (4.137)$$

$$= \sum_{\alpha} u_{\lambda'}^{\alpha}(x) u_{\lambda'}^{\alpha*}(x'). \quad (4.138)$$

eq4.151

So we can say that the sum over degeneracy of the bilinear product of the eigen function u_{λ}^{α} is proportional to the imaginary part of the Green's function.

4.10 Example: Infinite String

29 Jan p1
pr:InfStr1

Consider the case of an infinite string. In this case we take the end points $a \rightarrow -\infty$, $b \rightarrow \infty$ and the density σ , tension τ , and potential V as constants. The term V is the elastic constant of media.

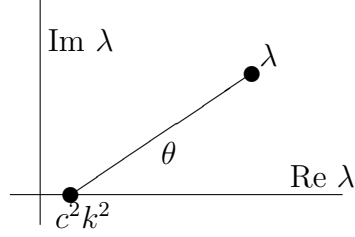
4.10.1 The Green's Function

In this case the Green's function is defined as the solution to the equation

$$\left[-\tau \frac{d^2}{dx^2} + V - \lambda\sigma \right] G(x, x'; \lambda) = \delta(x - x') \quad \text{for } -\infty < x, x' < \infty. \quad (4.139)$$

To get the solution we must take $\lambda (= \omega^2)$ to be imaginary. The solution for the Green's function can be written in terms of the normal modes (3.41)

$$G(x, x'; \lambda) = \frac{u_1(x_{<}) u_2(x_{>})}{-\tau W(u_1, u_2)} \quad (4.140)$$

Figure 4.8: θ specification.

where u_1 and u_2 satisfy the equation

$$\left[-\tau \frac{d^2}{dx^2} + V - \lambda \sigma \right] u_{1,2} = 0. \quad (4.141)$$

The boundary conditions are that u_1 is bounded and converges as $x \rightarrow -\infty$ and that u_2 is bounded and converges as $x \rightarrow \infty$. Divide both sides of equation 4.141 by $-\tau$ and substitute the definitions $\sigma/\tau = 1/c^2$ and $V/\tau = k^2$. This gives us the equation

$$\left[\frac{d^2}{dx^2} + \frac{\lambda - c^2 k^2}{c^2} \right] u(x) = 0. \quad (4.142)$$

The general solution to this equation can be written as

$$u(x) = A e^{i\sqrt{\lambda - c^2 k^2} x/c} + B e^{-i\sqrt{\lambda - c^2 k^2} x/c}. \quad (4.143)$$

We specify the root by the angle θ extending around the point $c^2 k^2$ on the real axis, as shown in figure 4.8. This is valid for $0 < \theta < 2\pi$. The correspondence of θ is as follows:

$$\lambda - c^2 k^2 = |\lambda - c^2 k^2| e^{i\theta}. \quad (4.144)$$

Thus

$$\sqrt{\lambda - c^2 k^2} = \sqrt{|\lambda - c^2 k^2|} e^{i\theta/2} \quad (4.145)$$

$$= \sqrt{|\lambda - c^2 k^2|} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right). \quad (4.146)$$

So that

$$\theta \rightarrow 0 \iff \sqrt{\lambda - c^2|k^2|} \quad (4.147)$$

$$\theta \rightarrow \pi \iff i\sqrt{\lambda - c^2|k^2|} \quad (4.148)$$

$$\theta \rightarrow 2\pi \iff -\sqrt{\lambda - c^2|k^2|}. \quad (4.149)$$

This is good everywhere except for values on the real line greater than c^2k^2 .

4.10.2 Uniqueness

29 Jan p2

The Green's function is unique since it was found using the theory of ordinary differential equations. We identify the fundamental solutions u_1, u_2 by looking at the large x behavior of 4.143.

$$e^{i\sqrt{\lambda - c^2k^2}x/c} \xrightarrow{x \rightarrow \infty} e^{-\infty} \quad \text{and} \quad e^{-i\sqrt{\lambda - c^2k^2}x/c} \xrightarrow{x \rightarrow \infty} e^{+\infty} \quad (4.150)$$

$$e^{i\sqrt{\lambda - c^2k^2}x/c} \xrightarrow{x \rightarrow -\infty} e^{+\infty} \quad \text{and} \quad e^{-i\sqrt{\lambda - c^2k^2}x/c} \xrightarrow{x \rightarrow -\infty} e^{-\infty} \quad (4.151)$$

so

$$u_1(x) = e^{-i\sqrt{\lambda - c^2k^2}\frac{x}{c}} \quad (4.152)$$

$$u_2(x) = e^{i\sqrt{\lambda - c^2k^2}\frac{x}{c}}. \quad (4.153)$$

The boundary condition has an explicit dependence on λ so the solution is not analytic. Notice that this time there is no way to get rid of the branch cut. The branch cut that comes in the solution is unavoidable because satisfaction of the boundary condition depends on the value of λ .

4.10.3 Look at the Wronskian

In the problem we are considering we have

$$W(u_1, u_2) = u_1 u_2' - u_2 u_1' \quad (4.154)$$

$$= +\frac{i}{c}\sqrt{\lambda - c^2k^2} - \left(-\frac{i}{c}\sqrt{\lambda - c^2k^2}\right) \quad (4.155)$$

$$= \frac{2i}{c}\sqrt{\lambda - c^2k^2}. \quad (4.156)$$

4.10.4 Solution

This gives the Greens' function

$$G(x, x'; \lambda) = \frac{ice^{-i\sqrt{\lambda-c^2k^2}x_{<}/c}e^{i\sqrt{\lambda-c^2k^2}x_{>}/c}}{2\tau\sqrt{\lambda-c^2k^2}} \quad (4.157)$$

$$= \frac{ic}{2\tau} \frac{e^{i\sqrt{\lambda-c^2k^2}|x-x'|/c}}{\sqrt{\lambda-c^2k^2}}. \quad (4.158)$$

We now have a branch cut for $\text{Re}(\lambda) \geq c^2k^2$ with branch point at $\lambda = c^2k^2$. Outside of this, G is analytic with no poles. This is analytic for λ in the cut λ -plane $\lambda > c^2k^2$. Consider the special case of large λ 29 Jan p3

$$G \xrightarrow{\lambda \rightarrow \infty} \frac{ic}{2\tau} \frac{e^{i\sqrt{\lambda}|x-x'|/c}}{\sqrt{\lambda}} \rightarrow 0. \quad (4.159)$$

Notice that this is the same asymptotic form we obtained in the discrete case when we looked at G as $\lambda \rightarrow \infty$ (see equation 4.79). Now consider the case $\lambda < c^2k^2$ on the real axis. This corresponds to the case that $\theta = \pi$. So

$$G(x, x'; \lambda) = \frac{c}{2\tau} \frac{e^{-\sqrt{c^2k^2-\lambda}|x-x'|}}{\sqrt{c^2k^2-\lambda}}. \quad (4.160)$$

This function is real and exponentially decreasing. For $\lambda > c^2k^2$ this function oscillates. If $\theta = 0$, it oscillates one way, and if $\theta = 2\pi$ it oscillates the other way, so the solution lacks uniqueness. The solutions correspond to different directions of traveling waves. eq4.159

4.10.5 Motivation, Origin of Problem

We want to understand the degeneracy in the Green's function for an infinite string. So we take a look at the physics behind the problem. How did the problem arise? It came from the time dependent problem with forced oscillation imposed by an impulsive force at x' :

$$\left[-\tau \frac{d^2}{dx^2} + V + \sigma \frac{\partial^2}{\partial t^2} \right] u(x, t) = \delta(x - x') e^{-i\omega t} \quad \text{where } \omega^2 < c^2k^2. \quad (4.161)$$

We wanted the steady state solution:

$$u(x, t) = e^{-i\omega t} G(x, x'; \lambda). \quad (4.162)$$

By substituting 4.160 this can be rewritten as

$$u(x, t) = \frac{c}{2\tau} \frac{e^{-i\omega t} e^{-\sqrt{c^2 k^2 - \omega^2} |x-x'|}}{-\sqrt{c^2 k^2 - \omega^2}}. \quad (4.163)$$

eq4.163ab

We consider the cases $\omega^2 < c^2 k^2$ and $\omega^2 > c^2 k^2$ separately.

pr:WaveProp1

If $\omega^2 < c^2 k^2$, then there is a unique solution and the exponential dies off. This implies that there is no wave propagation. This agrees with what the physics tells us intuitively: $k^2 c^2 > \omega^2$ implies large k , which corresponds to a large elastic constant V , which in turn means there will not be any waves.

29 Jan p4

For $\omega > c^2 k^2$ there is propagation of waves. For $\omega > c^2 k^2$ we denote the two solutions:

$$u_{\pm}(x, t) = e^{-i\omega t} G(x, x'; \lambda = \omega^2 \pm i\epsilon) \quad \text{as } \epsilon \rightarrow 0. \quad (4.164)$$

pr:cutoff2

As ω increases, it approaches the branch point. We define the cutoff frequency as being at the branch point, $\omega_c^2 = c^2 k^2$. We have seen that for $\omega < \omega_c$ there is no wave propagation, and for $\omega > \omega_c$ there is propagation, but we don't know the direction of propagation, so there is no unique solution.

The natural appearance of a branch cut with two solutions means that all the physics has not yet been given. We may rewrite equation 4.163 as:

$$u_{\pm}(x, t) = \frac{e^{-i\omega t \pm i(\sqrt{\omega^2 - c^2 k^2}/c)|x-x'|}}{\frac{2\tau}{c} \sqrt{\omega^2 - c^2 k^2}}. \quad (4.165)$$

These solutions to the steady state problem can be interpreted as follows. The solution u_+ represents a wave traveling to the right for points to the right of (i.e. on the positive side of) the source and a wave traveling to the left for points to the left of (i.e. on the negative side of) the source. Mathematically this means

$$u_+ = \begin{cases} \sim e^{-i\omega(t-x\sqrt{\omega^2-c^2k^2}/c\omega)} & \text{for } x > x' \\ \sim e^{-i\omega(t+x\sqrt{\omega^2-c^2k^2}/c\omega)} & \text{for } x < x'. \end{cases}$$

Similarly, the solution u_- represents a wave traveling to the left for points to the right of the source and a wave traveling to the right for points to the left of the source. Mathematically this means

$$u_- = \begin{cases} \sim e^{-i\omega(t+x\sqrt{\omega^2-c^2k^2}/c\omega)} & \text{for } x > x' \\ \sim e^{-i\omega(t-x\sqrt{\omega^2-c^2k^2}/c\omega)} & \text{for } x < x'. \end{cases}$$

These results can be rephrased by saying that u_+ is a steady state solution having only waves going out from the source, and u_- is a steady state solution having only waves going inward from the outside absorbed by the point. So the equation describes two situations, and the branch cut corresponds to the ambiguity in the situation.

4.11 Summary of the Infinite String

2 Feb p1

We have considered the equation

$$(L_0 - \lambda\sigma)G = \delta(x - x') \quad (4.166)$$

where

$$L_0 = -\tau \frac{d^2}{dx^2} + V. \quad (4.167)$$

We found that

is $V \neq 0$?

$$G(x, x'; \lambda) = \left(\frac{i}{2\tau\sqrt{\frac{\lambda}{c^2} - k^2}} \right) e^{i\sqrt{\frac{\lambda}{c^2} - k^2}|x-x'|} \quad (4.168)$$

where we have introduced the substitutions $V/\tau \equiv k^2$ and $\sigma/\tau \equiv 1/c^2$. The time dependent response is

$$u_{\pm}(x, t) = \frac{1}{2\tau\sqrt{\frac{\lambda}{c^2} - k^2}} e^{-i\omega t - i\sqrt{k^2 - \frac{\lambda}{c^2}}|x-x'|}. \quad (4.169)$$

If $\omega^2 < c^2k^2 \equiv \omega_c^2$ there is exponential decay, in which case there is no singularity of G at $\lambda = \omega^2$.

The other case is that

$$u_{\pm}(x, t) = e^{-i\omega t} G(x, x'; \lambda = \omega^2 \pm i\epsilon). \quad (4.170)$$

This case occurs when $\omega^2 > \omega_c^2$, for which there is a branch cut across the real axis. In this case we have traveling waves.

Note that in the case that $k = 0$ we always have traveling waves. The relevance of the equation $k^2 = V/\tau$ is that the resistance of the medium to propagation determines whether waves are produced. When $k = 0$ there is no static solution — wave propagation always occurs. If $\omega^2 < \omega_c^2$, then the period of the external force is small with respect to the response of the system, so that the media has no time to respond — the system doesn't know which way to go, so it exponentially decays.

Recall that the solution for the finite string (open or closed) allowed incoming and outgoing waves corresponding to reflections (for the open string) or different directions (for the closed string). Recall the periodic boundary condition problem:

$$u(x, t) = e^{-i\omega t} G(x, x'; \lambda) \quad (4.171)$$

$$= e^{-i\omega t} \frac{c}{2\omega} \frac{\cos[\frac{\omega}{c}(\frac{L}{2} - |x - x'|)]}{\sin[\frac{\omega}{c}\frac{L}{2}]} \quad (4.172)$$

This is equal to the combination of incoming and outgoing waves, which can be seen by expanding the cosine. We need the superposition to satisfy the boundary conditions and physically correspond to reflections at the boundaries. The sum of the two waves superimpose to satisfy the boundary conditions.

4.12 The Eigen Function Problem Revisited

1 Feb p3

We now return to the the connection with the eigen function problem.

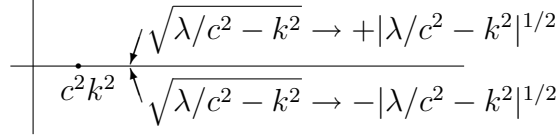
pr:efp2

We have seen before that the expression

$$\frac{G(\lambda = \lambda' + i\epsilon) - G(\lambda = \lambda' - i\epsilon)}{2\pi i} \quad (4.173)$$

vanishes if $\lambda' < c^2 k^2$. In the case that $\lambda' > c^2 k^2$ we have

$$\frac{G(\lambda = \lambda' + i\epsilon) - G(\lambda = \lambda' - i\epsilon)}{2\pi i} = \sum_{\alpha} u_{\lambda'}^{\alpha}(x) (u_{\lambda'}^{\alpha}(x'))^*. \quad (4.174)$$

Figure 4.9: Geometry in λ -plane

The geometry of this on the λ -plane is shown in figure 4.9. This gives us

$$\begin{aligned}
 & \frac{G(\lambda = \lambda' + i\epsilon) - G(\lambda = \lambda' - i\epsilon)}{2\pi i} \\
 &= \frac{1}{2\pi i} \frac{1}{2\tau \left| \frac{\lambda}{c^2} - k^2 \right|^{1/2}} (e^{i \left| \frac{\lambda}{c^2} - k^2 \right|^{1/2} |x-x'|} - e^{-i \left| \frac{\lambda}{c^2} - k^2 \right|^{1/2} |x-x'|}) \\
 &= \frac{1}{\pi} \frac{1}{2\tau \left| \frac{\lambda}{c^2} - k^2 \right|^{1/2}} \sin \left(\left| \frac{\lambda}{c^2} - k^2 \right|^{1/2} |x-x'| \right) \\
 &= \frac{1}{\pi} \text{Im} G(\lambda = \lambda' + i\epsilon).
 \end{aligned}$$

For $\lambda' > c^2 k^2$ we can write (using equation 4.138)

$$u_{\lambda'}^{\pm}(x) = \frac{1}{\sqrt{4\pi\tau \sqrt{\frac{\lambda'}{c^2} - k^2}}} e^{\pm i \sqrt{\lambda' - c^2 k^2} x/c} \quad (4.175)$$

for $\lambda' > c^2 k^2$. We now see that no eigen functions exist for $\lambda' < c^2 k^2$ since it exponentially increases as $\lambda' \rightarrow \infty$ and we must kill both terms. The Green's function is all right since $\lambda \in C$. eq4.188

4.13 Summary

1. For the Helmholtz equation, $\omega_n^2 > 0$, ω_n^2 is real, and the eigen functions are real.
2. The dispersion relation for a closed massless string with discrete mass points is

$$\omega_n^2 = \frac{c^2 \sin^2(k_n a/2)}{a^2/4}.$$

3. The Green's function obeys Hermitian analyticity:

$$G^*(x, x'; \lambda) = G(x', x; \lambda^*).$$

4. The form of the Green's function for λ near an eigen value λ_n is

$$G(x, x'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n} \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda}.$$

5. The Green's function for the fixed string problem is

$$G(x, x'; \lambda) = \frac{\sin \sqrt{\frac{\lambda}{c^2}} x_{<} \sin \sqrt{\frac{\lambda}{c^2}} (L - x_{>})}{\tau \sqrt{\frac{\lambda}{c^2}} \sin \sqrt{\frac{\lambda}{c^2}} L}.$$

6. The completeness relation is

$$\delta(x - x') = \sum_{n=1}^{\infty} \sigma(x) u_n(x) u_n^*(x).$$

7. The expansion theorem is

$$f(x) = \sum_{n=1}^{\infty} u_n(x) f_n$$

where

$$f_n = \int_a^b dx' u_n^*(x') \sigma(x') f(x').$$

8. The Green's function near the branch cut is related to the eigen functions by

$$\frac{1}{\pi} \text{Im} G(x, x'; \lambda' + i\epsilon) = \sum_{\alpha} u_{\lambda'}^{\alpha}(x) u_{\lambda'}^{\alpha}(x').$$

9. The Green's function solution for an infinite string is

$$G(x, x'; \lambda) = \frac{ic}{2\tau} \frac{e^{i\sqrt{\lambda - c^2 k^2} |x - x'|/c}}{\sqrt{\lambda - c^2 k^2}}.$$

4.14 References

The Rayleigh quotient is described in [Stakgold67a, p226ff] and [Stakgold79, p339ff].

For other ideas in this chapter, see Fetter and Stakgold.

A discussion of the discrete closed string is given in [Fetter80, p115].

The material in this chapter is also in [Fetter81, p245ff].

Chapter 5

Steady State Problems

Chapter Goals:

- Interpret the effect of an oscillating point source on an infinite string.
- Construct the Klein-Gordon equation and interpret its steady state solutions.
- Write the completeness relation for a continuous eigenvalue spectrum and apply it to the Klein-Gordon problem.
- Show that the solutions for the string problem with $\sigma = x$, $\tau = x$, and $V = m/x^2$ on the interval $0 < x < \infty$ are Bessel functions.
- Construct the Green's function for this problem.
- Construct and interpret the steady state solutions for this problem with a source point.
- Derive the Fourier-Bessel transform.

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few pages later

5.1 Oscillating Point Source

We now look at the problem with an oscillating point source. In the notation of the previous chapter this is

pr:ops1

3 Feb p1

$$\left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] u(x, t) = \delta(x - x') e^{-i\omega t} \quad -\infty < x, x' < \infty + \text{R.B.C.} \quad (5.1)$$

and can be written in terms of the Green function $G(x, x'; \lambda)$ which satisfies

$$[L_0 - \sigma(x)\lambda]G(x, x'; \lambda) = \delta(x - x') \quad -\infty < x, x' < \infty + \text{R.B.C.} \quad (5.2)$$

The steady state solution corresponding to energy radiated outward to infinity is

$$u(x, t) = e^{-i\omega t} G(x, x', \lambda = \omega^2 + i\epsilon). \quad (5.3)$$

The solution for energy radiated inward from infinity is the same equation with $\lambda = \omega^2 - i\epsilon$, but this is generally not a physical solution. This contrasts with the case of a finite region. In that case there are no branch cuts and there is no radiation.

1 Feb p5

5.2 The Klein-Gordon Equation

pr:kge1

We now apply the results of the previous chapter to another physical problem. Consider the equations of relativistic quantum mechanics. In the theory of relativity we have the energy relation

$$E^2 = m^2 c^4 + p^2 c^2 \quad (5.4)$$

In the theory of quantum mechanics we treat momentum and energy as operators

$$p \rightarrow -i\hbar \nabla \quad (5.5)$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad (5.6)$$

where

$$\dim[\hbar] = \text{Action} \quad (5.7)$$

We want to derive the appropriate wave equation, so we start with

$$E^2 - (m^2 c^4 + p^2 c^2) = 0 \quad (5.8)$$

Now substitute the operators into the above equation to get

$$\left[\left(i\hbar \frac{\partial}{\partial t} \right)^2 - \left(m^2 c^4 + \left(\frac{\hbar}{i} \nabla \right)^2 c^2 \right) \right] \Phi = 0 \quad (5.9)$$

This is the Klein–Gordon equation, which is a relativistic form of the Schrödinger Equation. Note that $|\Phi|^2$ still has a probability interpretation, as it does in non-relativistic quantum mechanics.

Now specialize this equation to one dimension.

1 Feb p6

$$\left[-\hbar^2 c^2 \frac{d^2}{dx^2} + m^2 c^4 + \hbar^2 \frac{\partial^2}{\partial t^2} \right] \Phi(x, t) = 0 \quad (5.10)$$

This is like the equation of a string (c.f., 1.11). In the case of the string the parameters were tension $\tau = \text{dim}[E/t]$, coefficient of elasticity $V = \text{dim}[E/l^3]$, and mass density $\sigma = [t^2 E/l^3]$, where E is energy, t is time, and l is length. The overall equation has units of force over length (since it is the derivative of Newton's second law). By comparing equation 1.11 with equation 5.10 we note the correspondence *eq5.energy*

$$V \rightarrow m^2 c^4 \quad \sigma \rightarrow \hbar^2 \quad \tau \rightarrow \hbar^2 c^2 \quad \text{and} \quad f(x, t) \rightarrow 0. \quad (5.11)$$

Note that $\sqrt{V/\tau} = mc/\hbar$ is a fundamental length known as the Compton wavelength, λ_c . It represents the intrinsic size of the particle. The expression $\sigma/\tau = 1/c^2$ shows that particle inertia corresponds to elasticity, which prevents the particle from responding quickly. Equation 5.10 has dimensions of energy squared (since it came from an energy equation).

We look again for steady state solutions

$$\Phi(x, t) = e^{-iE't/\hbar} \Phi_{E'}(x) \quad (5.12)$$

to get the eigen value problem

$$\left[-\hbar^2 c^2 \frac{d^2}{dx^2} + m^2 c^4 - E'^2 \right] \Phi_{E'}(x) = 0 \quad (5.13)$$

Thus we quote the previous result (equation 4.175 which solves equation 4.142)

$$\Phi_{E'}^{\pm}(x) = u^{\pm}(x) \quad (5.14)$$

where we let $k^2 \rightarrow (\frac{mc}{\hbar})^2$ and $\lambda' \rightarrow E'^2/\hbar^2$. As a notational shorthand, let $p' = \sqrt{E'^2 - m^2 c^4}/c$. Thus we write

$$\begin{aligned}\Phi_{E'}^{\pm}(x) &= \frac{e^{\pm i(\sqrt{E'^2 - m^2 c^4}/\hbar c)x}}{\sqrt{4\pi\hbar c\sqrt{E'^2 - m^2 c^4}}} \\ &= \frac{e^{\pm p'x/\hbar}}{\sqrt{4\pi\hbar p'c^2}}.\end{aligned}$$

The cut-off energy is mc^2 . We have the usual condition on the solution that $\lambda' > c^2 k^2$. This eigen value condition implies $E^2 > m^2 c^4$.

pr:rad1 In relativistic quantum mechanics, if $E < mc^2$, then no free particle is emitted at large distances. In the case that $E > mc^2$ there is radiation. Also note that m becomes inertia. For the case that $m \rightarrow 0$ there is always radiation. This corresponds to $V \rightarrow 0$ in the elastic string analogy. The potential V acts as an elastic resistance.

The Green's function has the form

$$G(x, x'; E) \sim \exp\{(\sqrt{m^2 c^4 - E^2}/\hbar c)|x - x'|\}.$$

So $G(x, x'; E)$ has a characteristic half-width of

$$|x - x'| \sim \hbar c / \sqrt{m^2 c^4 - E^2} = \hbar / p.$$

This is a manifestation of the uncertainty principle. As $x \rightarrow \infty$, for $E < mc^2$, the Green's function vanishes and no particle is radiated, while for $E > mc^2$ the Green's function remains finite at large distances which corresponds to the radiation of a particle of mass m .

5.2.1 Continuous Completeness

pr:CompRel3
3 Feb p2

Recall that the completeness condition in the discrete case is

$$\frac{\delta(x - x')}{\sigma(x)} = \sum_{\lambda', \alpha} u_{\lambda'}^{\alpha}(x) u_{\lambda'}^{\alpha*}(x'). \quad (5.15)$$

R. Horn says The corresponding equation for the case of continuous eigenvalues is no λ' in sum.

$$\frac{\delta(x - x')}{\sigma(x)} = \sum_{\lambda', \alpha=\pm} \int_{\omega_c}^{\infty} d\lambda' u_{\lambda'}^{\alpha}(x) u_{\lambda'}^{\alpha*}(x'). \quad (5.16)$$

Recall that in this case the condition for an eigen function to exist is $\lambda' > \omega_c^2$. In this case

$$u_{\lambda'}^{\pm}(x) = \frac{e^{\pm i \sqrt{\frac{\lambda'}{c^2} - k^2} x}}{\sqrt{4\pi\tau(\frac{\lambda'}{c^2} - k^2)^{1/2}}}. \quad (5.17)$$

Substituting the u^{\pm} into the continuous completeness relation gives

$$\delta(x - x') = \sigma(x) \frac{c}{4\pi\tau} \int_{\omega_c^2}^{\infty} \frac{d\lambda'}{(\frac{\lambda'}{c^2} - k^2)^{1/2}} \left[e^{i\sqrt{\lambda' - \omega_c^2}(\frac{x-x'}{c})} + e^{-i\sqrt{\lambda' - \omega_c^2}(\frac{x-x'}{c})} \right] \quad (5.18)$$

where $\omega_c^2 = k^2 c^2$. We now make a change of variables. We define the wave number as

$$\bar{k} = \frac{\sqrt{\lambda' - \omega_c^2}}{c} \quad (5.19)$$

It follows that the differential of the wave number is given by

$$d\bar{k} = \frac{1}{2c} \frac{d\lambda'}{\sqrt{\lambda' - \omega_c^2}}. \quad (5.20)$$

With this definition we can write

3 Feb p3

$$\delta(x - x') = \frac{2\sigma c^2}{4\pi\tau} \int_0^{\infty} d\bar{k} \left[e^{i\bar{k}(x-x')} + e^{-i\bar{k}(x-x')} \right]. \quad (5.21)$$

Note the symmetry of the transformation $\bar{k} \rightarrow -\bar{k}$. This property allows us to write

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\bar{k} e^{i\bar{k}(x-x')}. \quad (5.22)$$

This is a Fourier integral. In our problem this has a wave interpretation.

We now apply this to the quantum problem just studied. In this case $\lambda' = (E/\hbar)^2$ and $\omega_c^2 = m^2 c^4 / \hbar^2$. With these substitutions we can write

$$\bar{k} = \frac{(\frac{E^2}{\hbar^2} - \frac{m^2 c^4}{\hbar^2})^{1/2}}{c} \quad (5.23)$$

$$= \frac{1}{\hbar} \sqrt{\frac{E^2}{c^2} - m^2 c^2} = \frac{p}{\hbar} \quad (5.24)$$

so

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp' e^{i(x-x')p'/\hbar}. \quad (5.25)$$

pr:FI1

This Fourier integral has a particle interpretation.

5.3 The Semi-infinite Problem

3 Feb p4

Consider the following linear operator in the semi-infinite region

$$L_0 = -\frac{d}{dx} \left(x \frac{d}{dx} \right) + \frac{m^2}{x} \quad 0 < x < \infty. \quad (5.26)$$

Here we have let the string tension be $\tau = x$ and the potential be $V = m^2/x$. We will let the density be $\sigma(x) = x$. This is like a centrifugal potential. The region of consideration is $0 < x < \infty$.

This gives us the following Green's function equation (from 3.22)

$$\left[-\frac{d}{dx} \left(x \frac{d}{dx} \right) + \frac{m^2}{x} - \lambda x \right] G(x, x'; \lambda) = \delta(x - x') \quad (5.27)$$

defined on the interval $0 < x, x' < \infty$.

We now discuss the boundary conditions appropriate for the semi-infinite problem. We require that the solution be bounded at infinity, as was required in the infinite string problem. Note that in the above equation $\tau = 0$ at $x = 0$. Then at that point the right hand side of Green's second identity vanishes, as long as the amplitude at $x = 0$ is finite. Physically, $\tau \rightarrow 0$ at $x = 0$ means that the string has a free end. So there is a solution which becomes infinite at $x = 0$. This non-physical solution is eliminated by the boundary condition that the amplitude of that end is finite. Under these boundary conditions L_0 is hermitian for $0 \leq x < \infty$.

The solution can be written in the form (using 3.41)

$$G(x, x'; \lambda) = -\frac{u_1(x_<, \lambda)u_2(x_>, \lambda)}{xW(u_1, u_2)} \quad (5.28)$$

eq5.29

The function u_1 and u_2 are the solutions to the equation

$$(L_0 - \lambda x)u_{1,2} = 0 \quad (5.29)$$

where we restrict u_1 to be that function which is regular at $x = 0$, and u_2 to be that function which is bounded at infinity.

We note that the equations are Bessel's equations of order m : pr:Bessel1

$$\left[y^2 \frac{d^2}{dy^2} + y \frac{d}{dy} + (y^2 - m^2) \right] X_m(y) = 0 \quad (5.30)$$

where $y = x\sqrt{\lambda}$ and $X_m(y)$ is any solution. The solutions are then

$$u_1(x) = J_m(x\sqrt{\lambda}) \quad (5.31)$$

and

$$u_2(x) = H_m^{(1)}(x\sqrt{\lambda}). \quad (5.32)$$

The function $H_m^{(1)}(x\sqrt{\lambda})$ is known as the Hankel function. For large x 3 Feb p5 it may be approximated as

$$H_m^{(1)}(x\sqrt{\lambda}) \stackrel{x \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi x \sqrt{\lambda}}} e^{i(x\sqrt{\lambda} - \frac{m\pi}{2} - \frac{\pi}{4})} \quad (5.33)$$

and since $\text{Im}(\sqrt{\lambda}) > 0$, we have decay as well as out going waves.

Now we get the Wronskian:

Justify this

$$W(u_1, u_2) = W(J_m(x\sqrt{\lambda}), H_m^{(1)}(x\sqrt{\lambda})) \quad (5.34)$$

$$= iW(J_m(x\sqrt{\lambda}), N_m(x\sqrt{\lambda})) \quad (5.35)$$

$$= i\sqrt{\lambda} \left(\frac{2}{\pi x \sqrt{\lambda}} \right) = \frac{2i}{\pi x}. \quad (5.36)$$

In the second equality we used the definition $H_m^{(1)}(x) = J_m(x) + iN_m(x)$, where

$$N_m(x) \equiv \frac{J_m(x) \cos m\pi - J_{-m}(x)}{\sin m\pi}.$$

The third equality is verified in the problem set. So τW is independent of x , as expected.

Therefore (using equation 5.28)

$$G(x, x'; \lambda) = -\frac{1}{x} \frac{\pi x}{2i} J_m(x_{<}\sqrt{\lambda}) H_m^{(1)}(x_{>}\sqrt{\lambda}) \quad (5.37)$$

$$= \frac{i\pi}{2} J_m(x_{<}\sqrt{\lambda}) H_m^{(1)}(x_{>}\sqrt{\lambda}) \quad (5.38)$$

5.3.1 A Check on the Solution

Suppose that $\lambda < 0$. In this case we should have G real, since there is no branch cut. We have

$$\sqrt{\lambda} = i|\lambda|^{1/2} \quad \text{at} \quad \theta = \pi. \quad (5.39)$$

pr:Bessel2

We can use the definitions

$$I_m(x) \equiv e^{-im\pi/2} J_m(ix), \quad K_m(x) \equiv (\pi i/2) e^{im\pi/2} H_m^{(1)}(ix)$$

to write

$$G(x, x'; \lambda) = i \frac{\pi}{2} J_m(i|\lambda|^{1/2} x_<) H_m^{(1)}(i|\lambda|^{1/2} x_>) \quad (5.40)$$

$$= \frac{i\pi}{2} i^m I_m(x_< |\lambda|^{1/2}) \left(i^{-m} \frac{2}{\pi i} K_m(x_> |\lambda|^{1/2}) \right) \quad (5.41)$$

$$= I_m(x_< |\lambda|^{1/2}) K_m(x_> |\lambda|^{1/2}) \in R \quad (5.42)$$

Note also that $G \rightarrow 0$ as $x \rightarrow \infty$, so we have decay, and therefore no propagation. This is because asymptotically

$$I_m(z) \approx (2z\pi)^{-1/2} e^z \quad |\arg z| < 1/2, |z| \rightarrow \infty$$

$$K_m(z) \approx (2z/\pi)^{-1/2} e^{-z} \quad |\arg z| < 3\pi/2, |z| \rightarrow \infty.$$

5.4 Steady State Semi-infinite Problem

5 Feb p1

For the equation

$$\left[-\frac{d}{dx} \left(x \frac{d}{dx} \right) + \frac{m^2}{x} - \lambda x \right] G(x, x'; \lambda) = \delta(x - x') \quad \text{for } 0 < x, x' < \infty \quad (5.43)$$

the solution we obtained was

$$G(x, x'; \lambda) = \frac{i\pi}{2} J_m(\sqrt{\lambda} x_<) H_m^{(1)}(\sqrt{\lambda} x_>). \quad (5.44)$$

We now look at the steady state solution for the wave equation,

$$\left[L_0 + x \frac{\partial^2}{\partial t^2} \right] u(x, t) = \delta(x - x') e^{-i\omega t}. \quad (5.45)$$

We only consider outgoing radiation ($\omega^2 \rightarrow \omega^2 + i\varepsilon$). We take $\varepsilon > 0$, since $\varepsilon < 0$ corresponds to incoming radiation. So $\delta(x - x')$ acts as a point source, but not as a sink.

$$u(x, t) = e^{-i\omega t} G(x, x'; \lambda = \omega^2 + i\varepsilon) \quad (5.46)$$

$$= e^{-i\omega t} \frac{i\pi}{2} J_m(\omega x_<) H_m^{(1)}(\omega x_>) \quad (5.47)$$

$$= e^{-i\omega t} \frac{i\pi}{2} J_m(\omega x') H_m^{(1)}(\omega x_>) \quad \text{for } x > x'. \quad (5.48)$$

Next let $\omega x \gg 1$, so

$$u(x, t) = \frac{i\pi}{2} J_m(\omega x') \frac{e^{-i\omega(t-x)}}{\sqrt{\omega x}} \quad \text{for } \omega x \gg 1.$$

The condition $\omega x \gg 1$ allows us to use the asymptotic form of the Hankel function.

In this case we have outgoing (right moving) waves. These waves are composed of radiation reflected from the boundary $x = 0$ and from direct radiation. If in addition to $\omega x \gg 1$ we take $\omega x' \rightarrow 0$, then we have

$$u(x, t) = \frac{i\pi}{2} (\omega x')^m \frac{e^{-i\omega(t-x)}}{\sqrt{\omega x}} \quad (5.49)$$

We now look at the case $x < x'$ with ω large. In this case we have

$$\begin{aligned} u(x, t) &= e^{-i\omega t} \frac{i\pi}{2} H_m^{(1)}(\omega x') J_m(\omega x) \\ &= e^{-i\omega t} \frac{i\pi}{2} H_m^{(1)}(\omega x') \frac{1}{2} [H_m^{(1)}(\omega x) + H_m^{(2)}(\omega x)] \\ &\sim e^{-i\omega t} H_m^{(1)}(\omega x') \left[\frac{e^{i\omega x}}{\sqrt{\omega x}} + \frac{e^{-i\omega x}}{\sqrt{\omega x}} \right] \quad \omega x \gg 1 \\ &\sim H_m^{(1)}(\omega x') [e^{-i\omega(t-x)} + e^{-i\omega(t+x)}]. \end{aligned}$$

We now look at the Green's function as a complete set of eigenfunctions. First we consider

$$\frac{1}{2\pi i} [G(x, x'; \lambda' + i\varepsilon) - G(x, x'; \lambda' - i\varepsilon)]$$

$$\begin{aligned}
&= \frac{1}{2\pi i} [J_m(\sqrt{\lambda'}x_<)H_m^{(1)}(\sqrt{\lambda'}x_>) - J_m(-\sqrt{\lambda'}x_<)H_m^{(1)}(-\sqrt{\lambda'}x_>)] \\
&= \frac{1}{4} [J_m(\sqrt{\lambda'}x_>)] [H_n^{(1)}(\sqrt{\lambda'}x) + H_n^{(2)}(\sqrt{\lambda'}x)] \\
&= \frac{1}{2} J_m(\sqrt{\lambda'}x) J_m(\sqrt{\lambda'}x') \\
&= \frac{1}{\pi} \text{Im } G(x, x'; \lambda' + i\varepsilon).
\end{aligned}$$

5.4.1 The Fourier-Bessel Transform

The eigen functions $u_{\lambda'}$ satisfy

$$\left[\frac{-d}{dx} \left(x \frac{d}{dx} \right) + \frac{m^2}{x} - \lambda x \right] u_{\lambda'} = 0 \quad \text{for } 0 < x < \infty. \quad (5.50)$$

5 Feb p4

In this case since there is a boundary at the origin, waves move only to the right. There is no degeneracy, just one eigen function:

$$u_{\lambda'} = \sqrt{\frac{1}{2}} J_m(\sqrt{\lambda'}x). \quad (5.51)$$

We know from the general theory that if there is no degeneracy, then

$$\frac{1}{\sigma(x)} \delta(x - x') = \int_0^\infty d\lambda' u_{\lambda'}(x) u_{\lambda'}^*(x') \quad (5.52)$$

so

$$\frac{1}{x} \delta(x - x') = \frac{1}{2} \int_0^\infty d\lambda' J_m(\sqrt{\lambda'}x) J_m(\sqrt{\lambda'}x') \quad (5.53)$$

$$= \int_0^\infty \omega' d\omega' J_m(\omega'x) J_m(\omega'x'). \quad (5.54)$$

This is valid for $0 < x, x' < \infty$. Thus for $f(x)$ on $0 < x < \infty$ we have

$$f(x) = \int_0^\infty dx' f(x') \delta(x - x') \quad (5.55)$$

$$= \int_0^\infty dx' f(x') x' \int_0^\infty \omega' d\omega' J_m(\omega'x) J_m(\omega'x') \quad (5.56)$$

$$= \int_0^\infty \omega d\omega' J_m(\omega'x) \int_0^\infty dx' x' f(x') J_m(\omega'x'). \quad (5.57)$$

5 Feb p5

Thus for a given $f(x)$ on $0 < x < \infty$ we can write

$$f(x) = \int_0^\infty \omega' d\omega' J_m(\omega' x) F_m(\omega'). \quad (5.58)$$

This is the inversion theorem.

$$F_m(\omega) = \int_0^\infty x' dx' f(x') J_m(\omega x'). \quad (5.59)$$

This is the Fourier-Bessel transform of order m .

Explain why
this is useful?
pr:FBt1

5.5 Summary

1. The string equation of an oscillating point source on an infinite string has solutions corresponding to energy radiated in from or out to infinity.
2. The Klein-Gordon equation is

$$\left[-\hbar^2 c^2 \frac{d^2}{dx^2} + m^2 c^4 + \hbar^2 \frac{\partial^2}{\partial t^2} \right] \Phi(x, t) = 0.$$

Steady-state solutions for a point source with $|E| > mc^2$ correspond to a mass m particle radiated to (\pm) infinity, where as solutions with $|E| < mc^2$ die off with a characteristic range of $x \sim \hbar/p$.

pr:chRan1

3. The string problem with $\sigma = x$, $\tau = x$, and $V = m/x^2$ on the interval $0 < x < \infty$ corresponds the Bessel's equation

$$\left[y^2 \frac{d^2}{dy^2} + y \frac{d}{dy} + (y^2 - m^2) \right] X_m(y) = 0$$

where $y = x\sqrt{\lambda}$. The linearly independent pairs of solutions to this equation are the various Bessel functions: (i) $J_m(y)$ and $N_m(y)$, and (ii) $H_m^{(1)}(y)$ and $H_m^{(2)}(y)$.

4. The Green's function for this problem is

$$G(x, x'; \lambda) = \frac{i\pi}{2} J_m(\sqrt{\lambda} x_<) H_m^{(1)}(\sqrt{\lambda} x_>).$$

5. The steady state solutions for this problem with point source are

$$u(x, t) = e^{-i\omega t} \frac{i\pi}{2} J_m(\omega x_{<}) H_m^{(1)}(\omega x_{>}).$$

The outgoing solutions consist of direct radiation and radiation reflected from the $x = 0$ boundary.

6. The Fourier-Bessel transform is

$$F_m(\omega') = \int_0^\infty x' dx' f(x') J_m(\omega x').$$

The inversion theorem for this transform is

$$f(x) = \int_0^\infty \omega' d\omega' J_m(\omega' x) F_m(\omega').$$

5.6 References

The Green's function related to Bessel's equation is given in [Stakgold67a, p75].

Chapter 6

Dynamic Problems

Chapter Goals:

- State the problem which the retarded Green's function G_R solves, and the problem which the advanced Green's function G_A solves. Give a physical interpretation for G_R and G_A .
- Show how the retarded Green's function can be written in terms of the Green's function which solves the steady state problem.
- Find the retarded Green's function for an infinite string with σ and τ constant, and $V = 0$.
- Find the retarded Green's function for a semi-infinite string with a fixed end, σ and τ constant, and $V = 0$.
- Find the retarded Green's function for a semi-infinite string with a free end, σ and τ constant, and $V = 0$.

8 Feb p1
4 Jan p1
p1prv.yr.

- Explain how to find the retarded Green's function for an elastically bound semi-infinite string with σ and τ constant, and $V = 0$.
- Find an expression for the retarded Green's function in terms of the eigen functions.
- Show how the retarded boundary value problem can be restated as an initial value problem.

6.1 Advanced and Retarded GF's

pr:impf1

Consider an impulsive force, a force applied at a point in space along the string at an instant in time. This force is represented by

$$\sigma(x)f(x, t) = \delta(x - x')\delta(t - t'). \quad (6.1)$$

As with the steady state problem we considered in chapter 5, we apply no external forces on the boundary. If we can solve this problem, then we can solve the problem for a general time dependent force density $f(x, t)$.

We now examine this initial value problem (in contrast to the steady state problems considered in the previous chapter). We begin with the string at rest. Then we apply a blow at the point x' at the time t' . For this physical situation we want to find the solution

$$u(x, t) = G_R(x, t; x', t') \quad (6.2)$$

pr:GR1

where G_R stands for the retarded Green's function:

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x')\delta(t - t') \\ \text{for } a < x, x' < b; \text{ all } t, t'.$$

Now we look at the form of the two possible Regular Boundary Conditions. These two sets of conditions correspond to the case of an open and closed string. In the case of an open string the boundary condition is characterized by the equation

$$[\hat{n}_s \nabla + \kappa_s] G_R(x, t; x', t') = 0 \quad x \in S, a < x' < b; \forall t, t' \quad (6.3)$$

where S is the set of end points $\{a, b\}$. In the case of a closed string the boundary condition is characterized by the equations

$$G_R(x, t; x', t')|_{x=a} = G_R(x, t; x', t')|_{x=b} \quad \text{for } a < x' < b, \forall t, t', \quad (6.4)$$

$$\left. \frac{\partial}{\partial x} G_R(x, t; x', t') \right|_{x=a} = \left. \frac{\partial}{\partial x} G_R(x, t; x', t') \right|_{x=b} \quad \text{for } a < x' < b, \forall t, t'. \quad (6.5)$$

We now apply the condition that the string begins at rest:

8 Feb p2

$$G_R(x, t; x', t') = 0 \quad \text{for } t < t'. \quad (6.6)$$

This is called the retarded Green's function since the motionless string becomes excited as a result of the impulse. This cause-effect relationship is called *causality*. The RBC's are the same as in previous chapters but now apply to all times. pr:caus1

Another Green's function is G_A , which satisfies the same differential equation as G_R with RBC with the definition

$$G_A(x, t; x', t') = 0 \quad \text{for } t > t' \quad (6.7)$$

This is called the advanced Green's function since the string is in an excited state until the impulse is applied, after which it is at rest. In what follows we will usually be concerned with the retarded Green function, and thus write G for G_R (suppressing the R) except when contrasting the advanced and retarded Green functions. pr:AGF1

6.2 Physics of a Blow

We now look at the physics of a blow. Consider a string which satisfies the inhomogeneous wave equation with arbitrary force $\sigma(x)f(x)$. The momentum applied to the string, over time Δt , is then pr:momch1

$$\begin{aligned} \Delta p &= p(t_2) - p(t_1) \\ &= \int_{t_1}^{t_2} dt \frac{dp}{dt} \\ &= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \sigma(x) f(x, t). \end{aligned}$$

8 Feb p3

The third equality holds because dp/dt is the force, which in this case is $\int_{x_1}^{x_2} dx \sigma(x) f(x, t)$. We now look at the special case where the force is the δ function. In this case

$$\Delta p = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \delta(t - t') \delta(x - x') dx = 1 \quad (6.8)$$

for $x_1 < x' < x_2$, $t < t' < t_2$. Thus a delta force imparts one unit of momentum. Therefore we find that G_R is the response of our system to a localized blow at $x = x'$, $t = t'$ which imparts a unit impulse of momentum to the string.

6.3 Solution using Fourier Transform

We consider the Green's function which solves the following problem given by a differential equation and an initial condition:

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G(x, t; x', t') = \delta(t - t') \delta(x - x') \quad + \text{RBC}, \quad (6.9)$$

eq6de1

$$G(x, t; x', t') = 0 \quad \text{for } t < t'.$$

For fixed x we note that $G(x, t; x', t')$ is a function of $t - t'$ and not t and t' separately, since only $\partial^2/\partial t^2$ and $t - t'$ appear in the equation. Thus the transformation $t \rightarrow t + a$ and $t' \rightarrow t' + a$ does not change anything. This implies that the Green's function can be written

$$\begin{aligned} G(x, t; x', t') &= G(x, x'; t - t') \\ &= G(x, x'; \tau), \end{aligned}$$

where we define $\tau \equiv t - t'$. By this definition, $G = 0$ for $\tau < 0$.

pr:FTrans1

8 Feb p4

This problem can be solved by taking the complex Fourier transform:

$$\tilde{G}(x, x'; \omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G(x, x'; \tau). \quad (6.10)$$

eq6Ftran

Note that $\tilde{G}(x, x'; \omega)$ is convergent everywhere in the upper half ω -plane, which we now show. The complex frequency can be written as $\omega = \omega_R + i\omega_I$. Thus

$$e^{i\omega\tau} = e^{i\omega_R\tau} e^{-\omega_I\tau}.$$

For $\tilde{G}(x, x'; \omega)$ to exist, the integral must converge. Thus we require $e^{i\omega\tau} \rightarrow 0$ as $\tau \rightarrow \infty$, which means we must have $e^{-\omega_I\tau} \rightarrow 0$ as $\tau \rightarrow \infty$. This is only true when ω is in the upper half plane, $\omega_I > 0$. Thus \tilde{G} exists for all ω such that $\text{Im } \omega > 0$. Note that for \tilde{G}_A everything is reversed and ω is defined in the lower half plane.

By taking the derivative of both sides of $\tilde{G}(x, x'; \omega)$ in the Fourier transform, equation 6.10, we have

$$\frac{d}{d\omega} \tilde{G}(x, x'; \omega) = \int_{-\infty}^{\infty} d\tau \frac{d}{d\omega} e^{i\omega\tau} G(x, x'; \tau) = i \int_0^{\infty} d\tau \tau G(x, x'; \tau) e^{-i\omega\tau}.$$

which is finite. Therefore the derivative exists everywhere in the upper half ω -plane. Thus \tilde{G} is analytic in the upper half ω -plane. We have thus seen that the causality condition allows us to use the Fourier transform to show analyticity and pick the correct solution. The condition that $G = 0$ for $\tau < 0$ (causality) was only needed to show analyticity; it is not needed anymore.

Ask Baker
but how do we know
 $-i \int d\tau \tau G(\tau)$
converges?
8 Feb p5

We now Fourier transform the boundary condition of an open string 6.3:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} ((\hat{n}_S \cdot \nabla + \kappa_S) G) \\ &= (\hat{n}_S \cdot \nabla + \kappa_S) \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G \\ &= (\hat{n}_S \cdot \nabla + \kappa_S) \tilde{G}. \end{aligned}$$

Similarly, in the periodic case we regain the periodic boundary conditions of continuity $\tilde{G}_a = \tilde{G}_b$ and smoothness $\tilde{G}'_a = \tilde{G}'_b$. So \tilde{G} satisfies the same boundary conditions as G since the boundary conditions do not involve any time derivatives.

Consider equation 6.9 rewritten as

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G(x, x'; \tau) = \delta(x - x') \delta(\tau).$$

The Fourier transform of this equation is

$$L_0 \tilde{G}(x, x'; \omega) + \sigma(x) \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \frac{\partial^2}{\partial \tau^2} G(x, x'; \tau) = \delta(x - x'). \quad (6.11)$$

Using the product rule for differentiation, we can “pull out a divergence term”:

$$e^{i\omega\tau} \frac{\partial^2}{\partial \tau^2} G = \left(\frac{\partial^2}{\partial \tau^2} e^{i\omega\tau} \right) G + \frac{\partial}{\partial \tau} \left(e^{i\omega\tau} \frac{\partial}{\partial \tau} G - G \frac{\partial}{\partial \tau} e^{i\omega\tau} \right).$$

Thus our equation 6.11 becomes

$$\begin{aligned} \delta(x - x') &= L_0 \tilde{G}(x, x'; \omega) + \sigma(x)(-\omega^2 \tilde{G}(x, x'; \omega)) \\ &\quad + \sigma(x) \left[e^{i\omega\tau} \frac{\partial}{\partial t} G - G \frac{\partial}{\partial t} e^{i\omega\tau} \right]_{t=-\infty}^{t=+\infty}. \end{aligned}$$

Now we evaluate the surface term. Note that $G = 0$ for $\tau < 0$ implies $\partial G / \partial t = 0$, and thus $|_{-\infty} = 0$. Similarly, as $\tau \rightarrow \infty$, $e^{i\omega\tau} \rightarrow 0$ since $\text{Im } \omega > 0$, and thus $|^{\infty} = 0$. So we can drop the boundary term.

8 Feb p6

We thus find that G_R satisfies the differential equation

$$[L_0 - \omega^2 \sigma(x)] \tilde{G}(x, x'; \omega) = \delta(x - x') \quad \text{RBC}, \quad (6.12)$$

eq6stst

with $\text{Im } \omega > 0$. We now recognize that the Green function must be the same as in the steady state case:

$$\tilde{G}(x, x'; \omega) = G(x, x'; \lambda = \omega^2).$$

Recall that from our study of the steady state problem we know that the function $G(x, x'; \lambda)$ is analytic in the cut λ -plane. Thus by analytic continuation we know that $\tilde{G}(x, x'; \omega)$ is analytic in the whole cut plane. The convention $\omega = \sqrt{\lambda}$ compresses the region of interest to the upper half plane, where λ satisfies

this is unclear

$$[L_0 - \lambda \sigma(x)] G(x, x'; \lambda) = \delta(x - x') \quad + \text{RBC}, \quad (6.13)$$

eq6ststb

All that is left is to invert the Fourier transform.

6.4 Inverting the Fourier Transform

In the previous section we showed that for the Green's function G we have the Fourier Transform

$$\tilde{G}(x, x'; \omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(x, x'; \tau)$$

where $\tau = t - t'$, and we also found

$$\tilde{G}(x, x'; \omega) = G(x, x'; \lambda = \omega^2)$$

where $\omega = \omega_R + i\omega_I$ with $\omega_I > 0$, and $G(x, x'; \lambda = \omega^2)$ is the solution of the steady state problem. Now we only need to invert the Fourier Transform to get the retarded Green's function. We write

$$e^{i\omega\tau} = e^{i\omega_R\tau} e^{-\omega_I\tau}$$

so that

$$\underbrace{\tilde{G}(x, x'; \omega_R + i\omega_I)}_{\tilde{F}(\omega_R)} = \int_{-\infty}^{\infty} d\tau e^{i\omega_R\tau} \underbrace{[e^{-\omega_I\tau} G_R(x, x'; \tau)]}_{F(\tau)}.$$

This is a real Fourier Transform in terms of $F(\tau)$. We now apply the Fourier Inversion Theorem:

$$F(\tau) = e^{-\omega_I\tau} G_R(x, x'; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_R e^{-i\omega_R\tau} \tilde{G}(x, x'; \omega_R + i\omega_I)$$

so

$$G_R(x, x'; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_R e^{-i\omega_R\tau} \tilde{G}(x, x'; \omega) e^{\omega_I\tau} \quad (6.14)$$

$$\text{fix } \omega_I = \varepsilon \text{ and integrate over } \omega_R \quad (6.15)$$

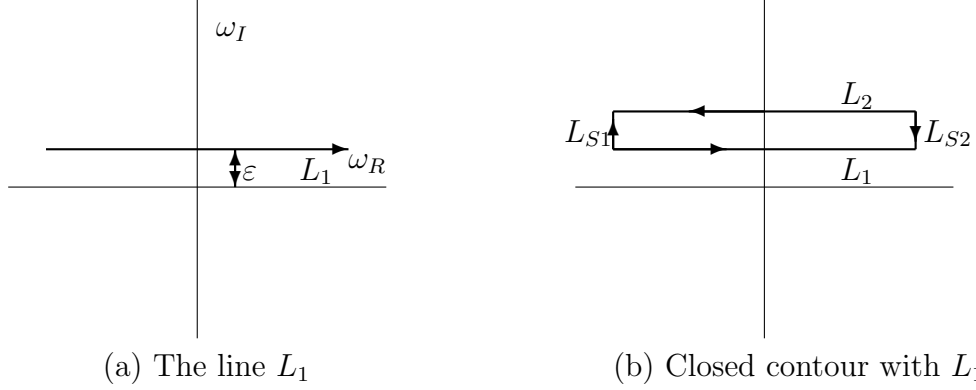
$$= \frac{1}{2\pi} \int_L d\omega e^{-i\omega\tau} \tilde{G}(x, x'; \omega) \quad (6.16)$$

where the contour L is a line in the upper half plane parallel to the ω_R axis, as shown in figure 6.1.a. The contour is off the real axis because of the branch cut. We note that any line in the upper half plane parallel to the real axis may be used as the contour of integration. This can be seen by considering the rectangular integral shown in figure 6.1.b. Because $\tilde{G} = 0$ as $\omega_R \rightarrow \infty$, we know that the sides L_{S1} and L_{S2} vanish. And since $e^{i\omega\tau}$ and \tilde{G} are analytic in the upper half plane, Cauchy's theorem tells use that the integral over the closed contour is zero. Thus the integrals over path L_1 and path L_2 must be equal.

but didn't
we use analytic
continuation to
ge the whole λ
plane.

pr:FIT1

10 Feb p2
fig6Lcont

Figure 6.1: The contour L in the λ -plane.

6.4.1 Summary of the General IVP

pr:IVP1

We have considered the problem of a string hit with a blow of unit momentum. This situation was described by the equation

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x')\delta(t - t') + \text{RBC} \quad (6.17)$$

eq6GFxt

with the condition that $G_R(x, t; x', t') = 0$ for $t < t'$. The Green's function which satisfies this equation was found to be

$$G_R(x, t; x', t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(x, x'; \omega). \quad (6.18)$$

eq6GFT

where $\tilde{G}(x, x'; \omega)$ satisfies the steady state Green's function problem.

6.5 Analyticity and Causality

To satisfy the physical constraints on the problem, we need to have $G_R = 0$ for $t < t'$. This condition is referred to as causality. This condition is obtained due to the fact that the product $e^{-i\omega(t-t')} \tilde{G}(x, x'; \omega^2)$ appearing in equation 6.17 is analytic. In this way we see that the analyticity of the solution allows it to satisfy the causality condition. As a check, for the case $t < t'$ we write $e^{-i\omega(t-t')} = e^{-i\omega_R(t-t')} e^{i\omega_I(t-t')}$ and close the contour as shown in figure 6.2. The quantity $e^{-i\omega(t-t')} \tilde{G}(x, x'; \omega^2)$

Need to show
somewhere
that $\tilde{G} \xrightarrow{|\omega| \rightarrow \infty} 0$.
fig6Luhp

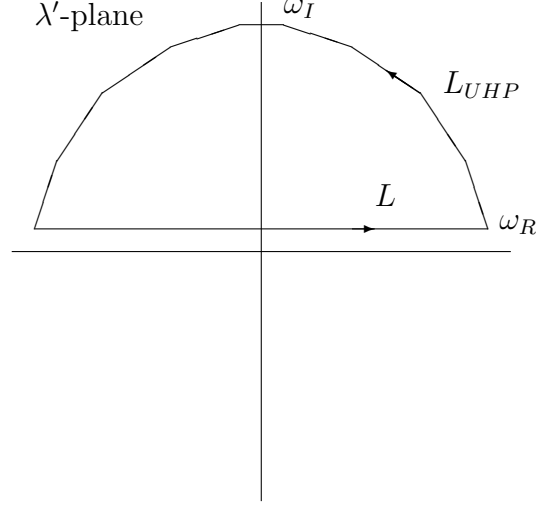


Figure 6.2: Contour $L_{C1} = L + L_{UHP}$ closed in UH λ -plane.

vanishes on the contour $L_{C1} = L + L_{UHP}$ since $e^{-i\omega_I(t-t')} \rightarrow 0$ as $\omega_I \rightarrow \infty$ and $|e^{-i\omega_R(t-t')}| = 1$, while we required $|\tilde{G}(x, x'; \omega^2)| \rightarrow 0$ as $|\omega| \rightarrow \infty$.

6.6 The Infinite String Problem

We now consider an infinite string where we take σ and τ to be a constant, and $V = 0$. Thus our linear operator (cf 1.10) is given by

pr:ISP1
10 Feb p3

$$L_0 = -\tau \frac{d^2}{dx^2}.$$

6.6.1 Derivation of Green's Function

We want to solve the equation

$$\left[-\tau \frac{\partial^2}{\partial x^2} + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x-x')\delta(t-t') \quad \text{for } -\infty < x, x' < \infty$$

(6.19)

with the initial condition

$$G_R(x, t; x', t') = 0 \quad \text{for } t < t'.$$

We write the Fourier transform of the Green's function in terms of λ :

$$\tilde{G}(x, x'; \omega) = G(x, x'; \lambda = \omega^2).$$

From 6.13, we know that $G(x, x'; \lambda)$ satisfies

$$\left[-\tau \frac{d^2}{dx^2} - \sigma \lambda \right] G(x, x'; \lambda) = \delta(x - x') \quad \text{for } -\infty < x, x' < \infty.$$

Actually since For the case of λ large we found the solution (4.159)
 $V = 0$

$$G = \frac{i}{2\sqrt{\lambda}} \frac{c}{\tau} e^{i\sqrt{\lambda/c^2}|x-x'|}$$

or ($\lambda \rightarrow \omega^2$)

$$\tilde{G}(x, x'; \omega) = \frac{i}{2\omega} \frac{c}{\tau} e^{i\omega|x-x'|/c}.$$

This gives us the retarded Green's function

$$G_R(x, t; x', t') = \frac{1}{2\pi} \int_L d\omega e^{-i\omega(t-t')} \frac{i}{2\omega} \frac{c}{\tau} e^{\frac{i\omega}{c}|x-x'|}. \quad (6.20)$$

eq6GRInfStr Now consider the term

$$e^{-i\omega \left[(t-t') - \frac{|x-x'|}{c} \right]}.$$

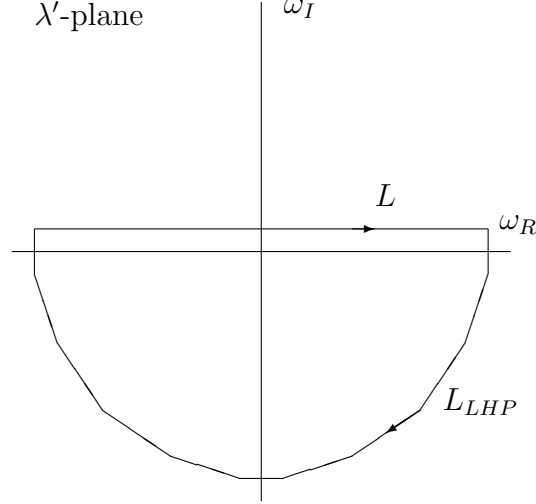
We treat this term in two cases:

- $t - t' < \frac{|x-x'|}{c}$. In this case

$$e^{-i\omega \left[(t-t') - \frac{|x-x'|}{c} \right]} \rightarrow 0 \quad \text{as } \omega_I \rightarrow \infty.$$

But the term $(i/2\omega) \rightarrow +\infty$ as $\omega_R \rightarrow 0$ in equation 6.20. Thus the integral vanishes along the contour L_{UHP} shown in figure 6.2 so (using Cauchy's theorem) equation 6.20 becomes

$$G_R(x, t; x', t') = \oint_L = \oint_{L+L_{UHP}} = 0.$$

Figure 6.3: Contour closed in the lower half λ -plane.

10 Feb p4

- $t - t' > \frac{|x-x'|}{c}$. In this case

$$e^{-i\omega \left[(t-t') - \frac{|x-x'|}{c} \right]} \rightarrow 0 \quad \text{as } \omega_I \rightarrow -\infty$$

fig6Llhp

so we close the contour below as shown in figure 6.3. Since the integral vanishes along L_{LHP} , we have $G_R = \int_L = \int_{L+L_{LHP}}$. Cauchy's theorem says that the integral around the closed contour is $-2\pi i$ times the sum of the residues of the enclosed poles. The only pole is at $\omega = 0$ and its residue is $\frac{1}{2\pi} \frac{ic}{2\tau}$. For this case we obtain

$$G_R(x, t; x', t') = -2\pi i \left(\frac{1}{2\pi} \frac{ic}{2\tau} \right) = \frac{c}{2\tau},$$

which is constant.

From these two cases we conclude that

$$G_R(x, t; x', t') = \frac{c}{2\tau} \theta \left(t - t' - \frac{|x - x'|}{c} \right). \quad (6.21)$$

The function θ is defined by the equation

$$\theta(u) = \begin{cases} 0 & \text{for } u < 0 \\ 1 & \text{for } u > 0. \end{cases}$$

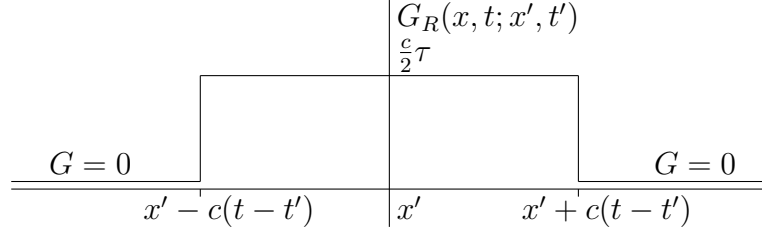


Figure 6.4: An illustration of the retarded Green's Function.

fig6retGF

The situation is illustrated in figure 6.4. This solution displays some interesting physical properties.

- The function is zero for $x < x' - c(t - t')$ and for $x > x' + c(t - t')$, so it represents an expanding pulse.
- The amplitude of the string is $c/2\tau$, which makes sense since for a smaller string tension τ we expect a larger transverse amplitude.

is this right?

- The traveling pulse does not damp out since $V = 0$.

6.6.2 Physical Derivation

10 Feb p5

We now explain how to get the solution from purely physical grounds. Consider an impulse Δp applied at position x' and time t' . Applying symmetry, at the first instant $\Delta p_y = 1/2$ for movement to the left and $\Delta p_y = 1/2$ for movement to the right. We may also write the velocity $\Delta v_y = \Delta p_y / \Delta m = \frac{1}{2} / \sigma dx$ since $\Delta p_y = 1/2$ and $\Delta m = \sigma dx$. By substituting $dx = c dt$, we find that in the time dt a velocity $\Delta v_y = 1/2 \sigma c dt$ is imparted to the string. This Δv_y is the velocity of the string portion at dx . By conservation of momentum, the previous string portion must now be stationary. In time dt the disturbance moves in the y direction an amount $\Delta y = v_y dt = \frac{1}{2\sigma c} = \frac{c}{2\tau}$. In these equalities we have used the identity $1/c^2 = \sigma/\tau$. Thus momentum is continually transferred from point to point (which satisfies the condition of conservation of momentum).

12 Feb p1

6.7 Semi-Infinite String with Fixed End

We now consider the problem of an infinite string with one end fixed. We will get the same form of Green's function. The defining equation is (c.f. 6.19)

$$\left[-\tau \frac{\partial^2}{\partial x^2} + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x') \delta(t - t') \quad (6.22)$$

for $-\infty < t, t' < \infty; 0 < x, x' < \infty$

with the further condition

$$G_R(x, t; x', t') = 0 \quad \text{for } x = 0. \quad (6.23)$$

This is called the Dirichlet boundary condition. We could use transform methods to solve this problem, but it is easier to use the *method of images* and the solution 6.21 to the infinite string problem.

To solve this problem we consider an infinite string with sources at x' and $-x'$. This gives us a combined force

$$\sigma f(x, t) = [\delta(x - x') - \delta(x + x')] \delta(t - t')$$

and the principle of superposition allows us to write the solution of the problem as the sum of the solutions for the forces separately:

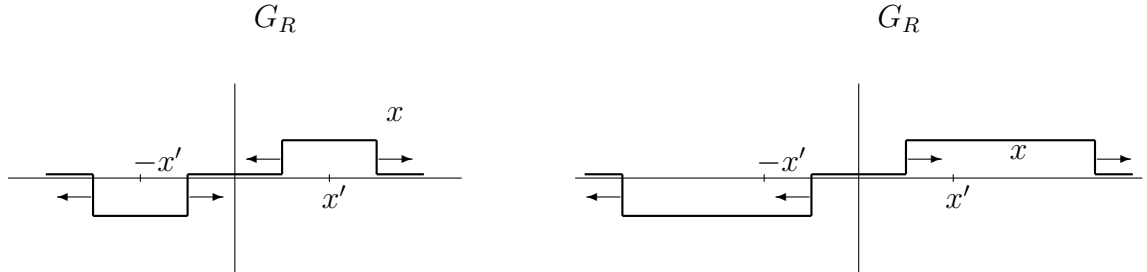
$$G_R(x, t; x', t') = \frac{c}{2\tau} \left[\theta \left(t - t' - \frac{|x - x'|}{c} \right) - \theta \left(t - t' - \frac{|x + x'|}{c} \right) \right] \quad (6.24)$$

where u is the solution (c.f. 6.21) of the infinite string with sources at x and x' . This solution is shown in figure 6.5. Since u satisfies 6.22 and 6.23, we can identify $G_R = u$ for $x \geq 0$. The case of a finite string leads to an infinite number of images to solve (c.f., section 8.7).

6.8 Semi-Infinite String with Free End

We now consider a new problem, that of a string with two free ends. The free end Green's function is

$$G_R = \frac{c}{2\tau} \left[\theta \left(t - t' - \frac{|x - x'|}{c} \right) + \theta \left(t - t' - \frac{|x + x'|}{c} \right) \right].$$

(a) G_R at time t (a) G_R at time t Figure 6.5: G_R at $t_1 = t' + \frac{1}{2}x'/c$ and at $t_2 = t' + \frac{3}{2}x'/c$.

This satisfies the equation

$$\left[-\tau \frac{\partial^2}{\partial x^2} + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x')\delta(t - t') \quad (6.25)$$

for $-\infty < t, t' < \infty$; $a < x, x' < b$

eq6qdblst2

with the boundary condition

$$\left. \frac{d}{dx} G_R(x, t; x', t') \right|_{x=0} = 0$$

which corresponds to $\kappa_a = 0$ and $h_a = 0$ is equation 6.3 (c.f., section 1.3.3).

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The derivative of $G_R(x = 0)$ is always zero. Note that

$$\int_a^b \frac{d}{dx} \theta(x) = \theta(b) - \theta(a) = \begin{cases} 1 & \text{for } a < 0 < b \\ 0 & \text{otherwise.} \end{cases} \quad (6.26)$$

which implies

$$\frac{d}{dx} \theta(x) = \delta(x).$$

But for any fixed $t - t'$ we can chose an ϵ such that the interval $[0, \epsilon]$ is is flat. Therefore

ask Baker

$$\frac{d}{dx} G_R^{\text{free end}} = 0.$$

Notes about the physics: For a string with a free end, the force on the end point is $F_y = \tau \frac{dG}{dx} = 0$ at $x = 0$ which implies $\frac{dG}{dx} = 0$ at $x = 0$ if the tension τ does not vanish. If the tension does vanish at $x = 0$, then we have a singular point at the origin and do not restrict $\frac{dG}{dx} = 0$ at $x = 0$.

6.9 Elastically Bound Semi-Infinite String

We now consider the problem with boundary condition

$$\left[-\frac{d}{dx} + \kappa \right] G_R = 0 \quad \text{for } x = 0.$$

The solution can be found using the standard transform method. Do an inverse Fourier transform of the Green's function in eq. 6.13 for the related problem $[-d/dx + \kappa]\tilde{G} = 0$. The frequency space part of this problem is done in problem 4.3.

6.10 Relation to the Eigen Fn Problem

We now look at the relation between the general problem and the eigen function problem (normal modes and natural frequencies). The normal mode problem is used in solving

$$[L_0 - \lambda\sigma]G(x, x'; \lambda) = \delta(x - x') \quad + RBC$$

for which $G(x, x'; \lambda)$ has poles at the eigen values of L_0 . We found in chapter 4 that $G(x, x'; \lambda)$ can be written as a bilinear summation

$$G(x, x'; \lambda) = \sum_n \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda} \quad (6.27)$$

where the $u_n(x)$ solve the normal mode problem:

eq6qA

$$[L_0 - \lambda_n\sigma]u_n(x) = 0 \quad + RBC. \quad (6.28)$$

Here we made the identification $\lambda_n = \omega_n^2$ where the ω_n 's are the natural *eq6qE*

frequencies and the u_n 's are the normal modes.

Recall also that the steady state solution for the force $\delta(x - x')e^{-i\omega t}$ is $u(x, t) = G(x, x'; \lambda = \omega^2 + i\varepsilon)e^{-i\omega t}$. The non-steady state response is $G_R(x, t; x', t')$ which is given by

$$G_R(x, t; x', t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(x, x'; \omega)$$

where $\tilde{G}(x, x'; \omega) = G(x, x'; \lambda = \omega^2)$. Plug $G(x, x'; \lambda)$ (from equation 6.27) into the Fourier transformed expression (equation 6.18). This gives

$$G_R(x, t; x', t') = \frac{1}{2\pi} \int_L d\omega e^{-i\omega(t-t')} \sum_n \frac{u_n(x)u_n^*(x')}{\lambda_n - \omega^2} \quad (6.29)$$

In this equation ω can be arbitrarily complex. (This equation is very different (c.f. section 4.6) from the steady state problem $G(x, x'; \lambda = \omega^2 + i\varepsilon)e^{-i\omega t}$ where ω was real.) Note that we are only interested in $t > t'$, since we have shown already that $G_R = 0$ for $t < t'$.

Now we add the lower contour since $e^{-i\omega(t-t')}$ is small for $t' > t$ and $\omega_I < 0$. This contour is shown in figure 6.3. The integral vanishes over the curved path, so we can use Cauchy's theorem to solve 6.29. The poles are at $\lambda_n = \omega^2$, or $\omega = \pm\sqrt{\lambda_n}$.

We now perform an evaluation of the integral for one of the terms of the summation in equation 6.29.

$$\begin{aligned} \int_{L+L_{LHP}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\lambda_n - \omega^2} &= - \int_{L+L_{LHP}} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{(\omega - \sqrt{\lambda_n})(\omega + \sqrt{\lambda_n})} \\ &= \frac{2\pi i}{2\pi} \left[\frac{e^{-i\sqrt{\lambda_n}(t-t')}}{2\sqrt{\lambda_n}} - \frac{e^{i\sqrt{\lambda_n}(t-t')}}{2\sqrt{\lambda_n}} \right] \\ &= \frac{\sin \sqrt{\lambda_n}(t-t')}{\sqrt{\lambda_n}}. \end{aligned} \quad (6.30)$$

By equation 6.29 we get

$$G_R(x, t; x', t') = \sum_n u_n(x)u_n^*(x') \frac{\sin \sqrt{\lambda_n}(t-t')}{\sqrt{\lambda_n}} \quad (6.31)$$

where $\sqrt{\lambda_n} = \omega_n$ and $t > t'$. This general solution gives the relationship between the retarded Green's function problem (equation 6.17) and the eigen function problem (eq. 6.28).

6.10.1 Alternative form of the G_R Problem

For $t - t'$ small, eq. 6.31 becomes

$$\begin{aligned} G_R(x, t; x', t') &\sim \sum_n \frac{u_n(x)u_n^*(x')}{\sqrt{\lambda_n}} \sqrt{\lambda_n}(t - t') \\ &= (t - t') \sum_n u_n(x)u_n^*(x') \\ &= (t - t') \frac{\delta(x - x')}{\sigma(x)}. \end{aligned}$$

Where we used the completeness relation 4.113. Thus for $t - t'$ small, 12 Feb 6 the G_R has the form

$$G_R(x, t; x', t')|_{t \rightarrow t'} \sim (t - t') \frac{\delta(x - x')}{\sigma(x)}.$$

Differentiating, this equation gives

$$\left. \frac{\partial}{\partial t} G_R(x, t; x', t') \right|_{t \rightarrow t'^+} = \frac{\delta(x - x')}{\sigma(x)}.$$

Also, as t approaches t' from the right hand side

$$G_R(x, t; x', t') = 0 \quad \text{for } t \rightarrow t'^-.$$

These results allow us to formulate an alternative statement of the G_R problem in terms of an initial value problem. The G_R is specified by the following three equations:

$$\left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = 0 \quad \text{for } t > t' + \text{RBC}$$

$$G_R(x, t; x', t') = 0 \quad \text{for } t = t'$$

$$\sigma(x) \frac{\partial}{\partial t} G_R = \delta(x - x') \quad \text{for } t = t'$$

Here $\sigma(x) \frac{\partial}{\partial t} G_R(x, t; x', t')$ represents a localized unit of impulse at x', t' (like $\Delta p = 1$). Thus we have the solution to the initial value problem for which the string is at rest and given a unit of momentum at t' .

We have now cast the statement of the G_R problem in two forms, as a retarded boundary value problem (RBVP) and as an initial value problem (IVP):

$$\begin{aligned} \text{RBVP} &= \begin{cases} \left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x') \delta(t - t') & + \text{RBC}, \\ G_R(x, t; x', t') = 0 & \text{for } t < t' \end{cases} \\ \text{IVP} &= \begin{cases} \left[L_0 + \sigma(x) \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = 0 & \text{for } t > t', \text{RBC}, \\ G_R(x, t; x', t') = 0 & \text{for } t = t', \\ \sigma(x) \frac{\partial}{\partial t} G_R = \delta(x - x') & \text{for } t = t'. \end{cases} \end{aligned}$$

6.11 Comments on Green's Function

17 Feb p1

6.11.1 Continuous Spectra

In the previous section we obtained the spectral expansion for discrete eigenvalues:

$$G_R(x, t; x', t') = \sum_{n=1}^{\infty} \frac{u_n(x) u_n^*(x')}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} (t - t') \quad (6.32)$$

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This gives us an expansion of the Green's function in terms of the natural frequencies.

For continuous spectra the sum is replaced by an integral

$$G_R = \int d\lambda_n \sum_{\alpha} u_{\lambda_n}^{\alpha} u_{\lambda_n}^{*\alpha} \frac{\sin \sqrt{\lambda_n} (t - t')}{\sqrt{\lambda_n}}$$

where we have included a sum over degeneracy index α (c.f. 4.108). Note that this result follows directly because the derivation in the previous section did not refer to whether we had a discrete or continuous spectrum.

6.11.2 Neumann BC

Recall that the RBC for an open string, equation 1.18, is

$$\left[-\frac{d}{dx} + \kappa_a \right] G_R = 0 \quad \text{for } x = a.$$

If $\kappa_a \rightarrow 0$ (Neumann boundary condition) then the boundary condition for the normal mode problem will be $(d/dx)u(x) = 0$ which will have a constant solution, i.e., $\lambda_1 = 0$. We cannot substitute $\lambda_1 = 0$ into equation 6.32, but instead must take the limit as λ_1 approaches zero. Physically, this corresponds to taking the elasticity κ_a as a small quantity, and then letting it go to zero. In this case we can write equation 6.32 with the λ_1 eigen value separated out:

$$\begin{aligned} G_R(x, t; x', t') &\xrightarrow{\lambda_1 \rightarrow 0} u_1(x)u_1^*(x')(t - t') \\ &\quad + \sum_n \frac{u_n(x)u_n^*(x')}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t - t') \\ &\xrightarrow{t \rightarrow \infty} u_1(x)u_1^*(x')(t - t'). \end{aligned} \quad (6.33)$$

Ask why

The last limit is true because the sum oscillates in t . The Green's function represents the response to a unit momentum, but $\kappa_a = 0$ which means there is no restoring force. Thus a change in momentum $\Delta p = 1$ is completely imparted to the string, which causes the string to acquire a constant velocity, so its amplitude increases linearly with time.

Note that equation 6.33 would still be valid if we had taken $\lambda_1 = 0$ in our derivation of the Green's function as a bilinear sum. In this case equation 6.30 would have a double pole for $\lambda_1 = 0$, so the residue would involve the derivative of the numerator, which would give the linear factor of $t - t'$.

Consider a string subject to an arbitrary force $\sigma(x)f(x, t)$. Remember that $\sigma(x)f(x, t) = \delta(t - t')\delta(x - x')$ gives $G_R(x, t; x', t')$. A general force $\sigma(x)f(x, t)$ gives a response $u(x, t)$ which is a superposition of Green's functions:

$$u(x, t) = \int_0^t dt' \int_a^b dx' G(x, t; x', t') \sigma(x') f(x', t')$$

with no boundary terms ($u(x, 0) = 0 = \frac{d}{dt}u(x, 0)$). Now plug in the Green's function expansion 6.33 to get

$$\begin{aligned} u(x, t) &= u_1(x) \int_0^t dt' (t - t') \int_a^b dx' u_1^*(x') \sigma(x') f(x', t') \\ &\quad + \sum_{n=2}^{\infty} \frac{u_n(x)}{\sqrt{\lambda_n}} \int_0^t dt' \sin \sqrt{\lambda_n}(t - t') \int_a^b dx' u_n^*(x') \sigma(x') f(x', t'). \end{aligned}$$

Note that again the summation terms oscillate with frequency ω_n . The spatial dependence is given by the $u_n(x)$. The coefficients give the projection of $\sigma(x)f(x, t)$ onto $u_n^*(x)$. In the $\lambda_1 = 0$ case the $u_1(x)$ term is constant.

6.11.3 Zero Net Force

Now let

$$F(t) \equiv (\text{const.}) \int dt' \sigma(x') f(x', t')$$

where $F(t')$ represents the total applied force at time t' . If $F(t') = 0$, then there are no terms which are linearly increasing in time contributing to the response $u(x, t)$. This is a meaningful situation, corresponding to a disturbance which sums to zero. The response is purely oscillatory; there is no growth or decay.

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not included
here

6.12 Summary

1. The retarded Green's function G_R solves

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] G_R(x, t; x', t') = \delta(x - x') \delta(t - t') + \text{RBC}$$

for $a < x, x' < b$; all t, t' .

with the condition

$$G_R(x, t; x', t') = 0 \quad \text{for } t < t'.$$

The advanced Green's function G_A solves the same equation, but with the condition

$$G_A(x, t; x', t') = 0 \quad \text{for } t > t'.$$

The retarded Green's function gives the response of the string (initially at rest) to a unit of momentum applied to the string at a point in time t' at a point x' along the string. The advanced Green's function gives the initial motion of the string such that a unit of momentum applied at x', t' causes it to come to rest.

2. The retarded Green's function can be written in terms of the steady state Green's function:

$$G_R(x, t; x' t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(x, x'; \omega).$$

3. The retarded Green's function for an infinite string with σ and τ constant, and $V = 0$ is

$$G_R(x, t; x', t') = \frac{c}{2\tau} \theta \left(t - t' - \frac{|x - x'|}{c} \right).$$

4. The retarded Green's function for a semi-infinite string with a fixed end, σ and τ constant, and $V = 0$ is

$$G_R(x, t; x', t') = \frac{c}{2\tau} \left[\theta \left(t - t' - \frac{|x - x'|}{c} \right) - \theta \left(t - t' - \frac{|x + x'|}{c} \right) \right].$$

5. The retarded Green's function for a semi-infinite string with a free end, σ and τ constant, and $V = 0$ is

$$G_R = \frac{c}{2\tau} \left[\theta \left(t - t' - \frac{|x - x'|}{c} \right) + \theta \left(t - t' - \frac{|x + x'|}{c} \right) \right].$$

6. The retarded Green's function can be written in terms of the eigen functions as

$$G_R(x, t; x', t') = \sum_n u_n(x) u_n^*(x') \frac{\sin \sqrt{\lambda_n}(t - t')}{\sqrt{\lambda_n}}.$$

6.13 References

A good reference is [Stakgold67b, p246ff].

This material is developed in three dimensions in [Fetter80, p311ff].

Chapter 7

Surface Waves and Membranes

Chapter Goals:

- Show how the equation describing shallow water surface waves is related to our most general differential equation.
- Derive the equation of motion for a 2-dimensional membrane and state the corresponding regular boundary conditions.

7.1 Introduction

In this chapter we formulate physical problems which correspond to equations involving more than one dimension. This serves to motivate the mathematical study of N -dimensional equations in the next chapter.

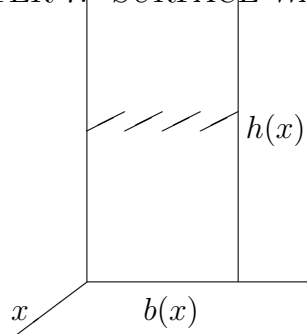


Figure 7.1: Water waves moving in channels.

7.2 One Dimensional Surface Waves on Fluids

7.2.1 The Physical Situation

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fig:7.1
pr:hww1

Consider the physical situation of a surface wave moving in a channel¹. This situation is represented in figure 7.1. The height of equilibrium is $h(x)$ and the width of the channel is $b(x)$. The height of the wave $z(x, t)$ can then be written as

$$z(x, t) = h(x) + u(x, t)$$

where $u(x, t)$ is the deviation from equilibrium. We now assume the shallow wave case $u(x, t) \ll h(x)$. This will allow us to linearize the Navier–Stokes equation.

7.2.2 Shallow Water Case

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This is the case in which the height satisfies the condition $h(x) \ll \lambda$ where λ is the wavelength. In this case the motion of the water is approximately horizontal. Let $S(x) = h(x)b(x)$. The equation of continuity and Newton’s law (i.e., the Navier–Stokes equation) then give

$$-\frac{\partial}{\partial x} \left(gS(x) \frac{\partial}{\partial x} \right) u + b(x) \frac{\partial^2}{\partial t^2} u(x, t) = 0,$$

¹This material corresponds to FW p. 357–363.

which is equivalent to the 1-dimensional string, where $\sigma(x) \Rightarrow b(x)$ and $\tau(x) \Rightarrow gS(x)$. 17 Feb p4

Consider the case in which $b(x)$ is independent of x :

$$-\frac{\partial}{\partial x} \left(gh(x) \frac{\partial}{\partial x} \right) u + \frac{\partial^2}{\partial t^2} u(x, t) = 0. \quad (7.1)$$

This corresponds to $\sigma = 1$ and $\tau = gh(x)$.

eq7shallow

Propagation of shallow water waves looks identical to waves on a string. For example, in problem 3.5, $h(x) = x$ gives the Bessel's equation, with the identification $\tau(x) = x$ and $V(x) = m^2/x$.

As another example, take $h(x)$ to be constant. This gives us wave propagation with $c = \sqrt{\tau/\sigma}$, $\tau = gh$, and $\sigma = 1$. So the velocity of a water wave is $c = \sqrt{gh}$. The deeper the channel, the faster the velocity. This partially explains wave breaking: The crest sees more depth than the trough.

7.3 Two Dimensional Problems

17 Feb p5

We now look at the 2-dimensional problem, that of an elastic membrane².

We denote the region of the membrane by R and the perimeter (1- dimensional “surface”) by S . The potential energy differential for an element of a 1-dimensional string is pr:elmem1

$$dU = \frac{1}{2} \tau(x) \left(\frac{du}{dx} \right)^2 dx.$$

In the case of a 2-dimensional membrane we replace $u(x)$ with $u(x, y) = u(\mathbf{x})$. In this case the potential energy difference is (see section 2.4.2)

$$dU = \frac{1}{2} \tau(x, y) \left(\left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right) dx dy \quad (7.2)$$

$$= \frac{1}{2} \tau(\mathbf{x}) (\nabla u)^2 d\mathbf{x} \quad (7.3)$$

²This is discussed on p. 271–288 of FW.

where τ is the tension of the membrane. Note that there is no mixed term $\frac{d}{dx}\frac{d}{dy}u$ since the medium is homogeneous. The total potential energy is given by the equation

$$U = \int_R d\mathbf{x} \frac{1}{2} \tau(\mathbf{x}) (\nabla u)^2,$$

where $\tau(\mathbf{x})$ is the surface tension. In this equation $d\mathbf{x} = dx dy$ and $(\nabla u)^2 = (\nabla u) \cdot (\nabla u)$. The total kinetic energy is

$$T = \frac{1}{2} m v^2 = \int_R d\mathbf{x} \frac{1}{2} \sigma(\mathbf{x}) \left(\frac{\partial u}{\partial t} \right)^2.$$

Now think of the membrane as inserted in an elastic media. We then get an addition to the $U(\mathbf{x})$ energy due to elasticity, $\frac{1}{2} V(\mathbf{x}) u(\mathbf{x}, t)^2$. We also add an additional force which will add to the potential energy:

$$f(\mathbf{x}, t) \sigma(\mathbf{x}) d\mathbf{x} u(\mathbf{x}, t) = \left(\frac{\text{force}}{\text{mass}} \right) \left(\frac{\text{mass}}{\text{length}} \right) (\text{length}) (\text{displacement}).$$

pr:lagr1

The Lagrangian is thus

$$L = \frac{1}{2} \int_R d\mathbf{x} \times \left[\sigma \mathbf{x} \left(\frac{\partial}{\partial t} \mathbf{u} \right)^2 - \tau(\mathbf{x}) (\nabla u)^2 - V(\mathbf{x}) u(\mathbf{x})^2 - f(\mathbf{x}, t) \sigma(\mathbf{x}) d\mathbf{x} u(\mathbf{x}, t) \right].$$

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Notice the resemblance of this Lagrangian to the one for a one dimensional string (see section 2.4.2).

19 Feb p2

We apply Hamiltonian Dynamics to get the equation of motion:

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \sigma(\mathbf{x}) f(\mathbf{x}, t)$$

where $L_0 = -\nabla(\tau(\mathbf{x})\nabla) - V(\mathbf{x})$. This is identical to the equation of motion for a string except now in two dimensions. It is valid everywhere for \mathbf{x} inside the region R .

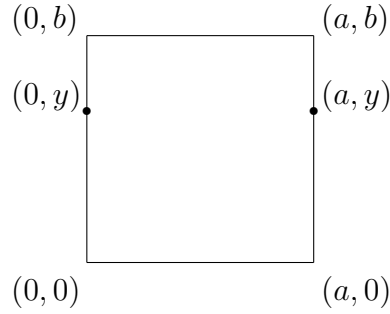


Figure 7.2: The rectangular membrane.

7.3.1 Boundary Conditions

Elastically Bound Surface

The most general statement of the boundary condition for an elastically bound surface is

$$[\hat{n} \cdot \nabla + \kappa(x)]u(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{for } x \text{ on } S. \quad (7.4)$$

In this equation the “surface” S is the perimeter of the membrane, \hat{n} is the outward normal for a point on the perimeter, $\kappa(\mathbf{x}) = k(\mathbf{x})/\tau(\mathbf{x})$ is the effective spring constant at a point on the boundary, and $h(\mathbf{x}, t) = f(\mathbf{x}, t)/\tau(\mathbf{x})$ is an external force acting on the boundary S .

Periodic Boundary Conditions

We now consider the case of a rectangular membrane, illustrated in figure 7.2, with periodic boundary conditions:

$$\begin{aligned} u(0, y) &= u(a, y) & \text{for } 0 \leq y \leq b, \\ u(x, 0) &= u(x, b) & \text{for } 0 \leq x \leq a, \end{aligned} \quad (7.5)$$

and

$$\left. \frac{\partial^2}{\partial x^2} u(x, y) \right|_{x=0} = \left. \frac{\partial^2}{\partial x^2} u(x, y) \right|_{x=a} \quad \text{for } 0 \leq y \leq b,$$

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pr:pb2

fig:7.2

$$\left. \frac{\partial^2}{\partial x^2} u(x, y) \right|_{y=0} = \left. \frac{\partial^2}{\partial x^2} u(x, y) \right|_{y=b} \quad \text{for } 0 \leq x \leq a.$$

In this case we can consider the region R to be a torus.

7.4 Example: 2D Surface Waves

19 Feb p4

We now give one last 2-dimensional example³. We consider a tank of water whose bottom has arbitrary height $h(\mathbf{x})$ and look at the surface waves. This example connects the 1-dimensional surface wave problem and the 2-dimensional membrane problem.

For this problem the vertical displacement is given by

$$z(\mathbf{x}, t) = h(\mathbf{x}) + u(\mathbf{x}, t),$$

with $\lambda \gg h(\mathbf{x})$ for the shallow water case and $u \ll h(\mathbf{x})$. Thus (using 7.1) our equation of motion is

$$\left[-\nabla \cdot (gh(\mathbf{x})\nabla) + \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = f(\mathbf{x}, t).$$

Note that in this equation we have $\sigma = 1$. For this problem we take the Neumann natural boundary condition:

$$\hat{n} \cdot \nabla u(\mathbf{x}, t) = 0$$

and

$$\frac{\partial}{\partial t} u_{\perp} = -g \nabla_{\perp} u|_S$$

for x on S . This is the case of rigid walls. The latter equation just means that there is no perpendicular velocity at the surface.

The case of membranes for a small displacement is the same as for surface waves. We took $\sigma = 1$ and $\tau(\mathbf{x}) = gh(\mathbf{x})$.

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The formula for all these problems is just

$$\left[L_0 + \sigma(\mathbf{x}) \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \sigma(\mathbf{x}) f(\mathbf{x}, t) \quad \text{for } \mathbf{x} \text{ in } R$$

where $L_0 = -\nabla(\tau(\mathbf{x})\nabla) + V(\mathbf{x})$. (Note that $\tau(\mathbf{x})$ is not necessarily tension.) We let $\mathbf{x} = (x_1, \dots, x_n)$ and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$. The boundary conditions can be elastic or periodic.

³This one comes from FWp. 363–366.

7.5 Summary

1. The equation for shallow water surface waves and the equation for string motion are the same if we identify gravity times the cross-sectional area with “tension”, and the width of the channel with “mass density”.
2. The two-dimensional membrane problem is characterized by the wave equation

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \sigma(\mathbf{x}) f(\mathbf{x}, t)$$

where $L_0 = -\nabla(\tau(\mathbf{x})\nabla) - V(\mathbf{x})$, subject to either an elastic boundary condition,

$$[\hat{n} \cdot \nabla + \kappa(x)]u(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{for } x \text{ on } S,$$

or a periodic boundary condition.

7.6 References

The material on surface waves is covered in greater depth in [Fetter80, p357ff], while the material on membranes can be found in [Fetter80, p271].

Chapter 8

Extension to N -dimensions

Chapter Goals:

- Describe the different sorts of boundaries and boundary conditions which can occur for the N -dimensional problem.
- Derive the Green's identities for the N -dimensional case.
- Write the solution for the N -dimensional problem in terms of the Green's function.
- Describe the method of images.

22 Feb p1

8.1 Introduction

In the previous chapter we obtained the general equation in two dimensions:

$$\left[L_0 + \sigma(\mathbf{x}) \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \sigma(\mathbf{x}) f(\mathbf{x}, t) \quad \text{for } \mathbf{x} \text{ in } R \quad (8.1)$$

where

$$L_0 = -\nabla \cdot \tau(\mathbf{x}) \nabla + V(x).$$

eq8usf

This is immediately generalizable to N -dimensions. We simply let $\mathbf{x} = (x_1, \dots, x_n)$ and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$. and we introduce the

notation $\hat{n} \cdot \nabla u \equiv \partial u / \partial n$. R is now a region in N -dimensional space and S is the $(N - 1)$ -dimensional surface of R .

pr:bc3

The boundary conditions can either be elastic or periodic:

1. *Elastic*: The equation for this boundary condition is

$$[\hat{n} \cdot \nabla + K(\mathbf{x})]u(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{for } \mathbf{x} \text{ on } S. \quad (8.2)$$

eq8.1b

The term $K(\mathbf{x})$ is like a spring constant which determines the properties of the medium on the surface, and $h(\mathbf{x}, t)$ is an external force on the boundary. These terms determine the outward gradient of $u(\mathbf{x}, t)$.

pr:pb3

2. *Periodic*: For the two dimensional case the region looks like 7.2 and the periodic boundary conditions are 7.5 and following. In the N -dimensional case the region R is an N -cube. Connecting matching periodic boundaries of S yields an N -torus in $(N + 1)$ -dimensional space.

To uniquely specify the time dependence of $u(\mathbf{x}, t)$ we must specify the initial conditions

$$u(\mathbf{x}, t)|_{t=0} = u_0(\mathbf{x}) \quad \text{for } \mathbf{x} \text{ in } R \quad (8.3)$$

$$\frac{\partial}{\partial t}(\mathbf{x}, t)|_{t=0} = u_1(\mathbf{x}) \quad \text{for } \mathbf{x} \text{ in } R. \quad (8.4)$$

For the 1-dimensional case we solved this problem using Green's Identities. In section 8.3 we will derive the Green's Identities for N -dimensions.

22 Feb p2

8.2 Regions of Interest

There are three types of regions of interest: the Interior problem, the Exterior problem, and the All-space problem.

pr:intprob1

1. *Interior problem*: Here R is enclosed in a finite region bounded by S . In this case we expect a discrete spectrum of eigenvalues, like one would expect for a quantum mechanical bound state problem or for pressure modes in a cavity.

pr:extprob1

2. *Exterior problem:* Here R extend to infinity in all directions but is excluded by a finite region bounded by S . In this case we expect a continuum spectrum if $V > 0$ and a mixed spectrum if $V < 0$. This is similar to what one would expect for quantum mechanical scattering.
3. *All-space problem:* Here R extends to infinity in all directions and is not excluded from any region. This can be considered a degenerate case of the Exterior problem.

pr:allprob1

8.3 Examples of N -dimensional Problems

8.3.1 General Response

pr:GenResp2

In the following sections we will show that the N -dimensional general response problem can be solved using the Green's function solution to the steady state problem. The steps are identical to those for the single dimension case covered in chapter 6.

$$\begin{aligned}
 G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2) &= \tilde{G}(\mathbf{x}, \mathbf{x}'; \lambda) \\
 &\rightarrow G_R(\mathbf{x}, t; \mathbf{x}', t') \quad \text{retarded} \\
 &\rightarrow u(\mathbf{x}, t) \quad \text{General Response.}
 \end{aligned}$$

8.3.2 Normal Mode Problem

pr:NormMode3

The normal mode problem is given by the homogeneous differential equation

$$\left(L_0 + \sigma(\mathbf{x}) \frac{\partial^2}{\partial t^2} \right) u(\mathbf{x}, t) = 0.$$

Look for solutions of the form

$$u(\mathbf{x}, t) = e^{-i\omega_n t} u_n(\mathbf{x}).$$

The natural frequencies are $\omega_n = \sqrt{\lambda_n}$, where λ_n is an eigen value. The normal modes are eigen functions of L_0 . Note: we need RBC to ensure that L_0 is Hermitian.

24 Feb p2
pr:fhop2

8.3.3 Forced Oscillation Problem

The basic problem of steady state oscillation is given by the equation

$$\left(L_0 + \sigma \frac{\partial^2}{\partial t^2}\right) u(\mathbf{x}, t) = e^{-i\omega t} \sigma(\mathbf{x}) f(\mathbf{x}) \quad \text{for } \mathbf{x} \in R$$

and (for example) the elastic boundary condition

$$(\hat{n} \cdot \nabla + K)u = h(\mathbf{x})e^{-i\omega t} \quad \text{for } \mathbf{x} \in S.$$

We look for steady state solutions of the form

$$u(\mathbf{x}, t) = e^{-i\omega t} u(\mathbf{x}, \omega).$$

The value of ω is chosen, so this is not an eigen value problem. We assert

$$\begin{aligned} u(\mathbf{x}, \omega) &= \int_{\mathbf{x}' \in R} d\mathbf{x}' G(\mathbf{x}, \mathbf{x}'; \lambda = \omega + i\varepsilon) \sigma(\mathbf{x}') f(\mathbf{x}') \\ &\quad + \int_{\mathbf{x}' \in S} d\mathbf{x}' \tau(\mathbf{x}') G(\mathbf{x}, \mathbf{x}'; \lambda = \omega + i\varepsilon) \sigma(\mathbf{x}) h(\mathbf{x}'). \end{aligned}$$

The first term gives the contribution due to forces on the volume and the second term gives the contribution due to forces on the surface.

In the special case that $\sigma(\mathbf{x})f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$, we have

$$u(\mathbf{x}, t) = e^{-i\omega t} G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2).$$

8.4 Green's Identities

In this section we will derive Green's 1st and 2nd identities for the N -dimensional case. We will use the general linear operator for N -dimensions

$$L_0 = -\nabla \cdot (\tau(\mathbf{x})\nabla) + V(\mathbf{x})$$

and the inner product for N -dimensions

$$\langle S, L_0 u \rangle = \int d\mathbf{x} S^*(\mathbf{x}) L_0 u(\mathbf{x}). \quad (8.5)$$

eq8tst

8.4.1 Green's First Identity

The derivation here generalizes the derivation given in section 2.1.

$$\begin{aligned}
 \langle S, L_0 u \rangle &= \int_R d\mathbf{x} S^*(\mathbf{x}) [-\nabla \cdot (\tau(\mathbf{x}) \nabla) + V(\mathbf{x})] u(\mathbf{x}) \\
 &\quad \text{integrate 1st term by parts} \\
 &= \int_R d\mathbf{x} [-\nabla \cdot (S^* \tau(\mathbf{x}) \nabla u) + (\nabla S^*) \tau(\mathbf{x}) \nabla u + S^* V u] \\
 &\quad \text{integrate 1st term using Gauss' Theorem} \\
 &= - \int_S dS \hat{n} \cdot (S^* \tau(\mathbf{x}) \nabla u) + \int_R [S^* V u + (\nabla \cdot S^* \tau(\mathbf{x}) \nabla u)] d\mathbf{x}
 \end{aligned} \tag{8.6}$$

This is Green's First Identity generalized to N -dimensions:

$$\langle S, L_0 u \rangle = - \int_S dS \hat{n} \cdot (S^* \tau(\mathbf{x}) \nabla u) + \int_R [S^* V u + (\nabla \cdot S^* \tau(\mathbf{x}) \nabla u)] d\mathbf{x}. \tag{8.7}$$

Compare this with 2.3.

8.4.2 Green's Second Identity

We now interchange S and u . In the quantity $\langle S, L_0 u \rangle - \langle L_0 S, u \rangle$ the symmetric terms will drop out, i.e., the second integral in 8.7 is cancelled. We are left with

$$\langle S, L_0 u \rangle - \langle L_0 S, u \rangle = \int_S dS \hat{n} \cdot [-S^* \tau(\mathbf{x}) \nabla u + u \tau(\mathbf{x}) \nabla S^*].$$

8.4.3 Criterion for Hermitian L_0

If u, S^* satisfy the RBC, then the surface integral in Green's second identity vanishes. This leaves $\langle S, L_0 u \rangle - \langle L_0 S, u \rangle$, which means that L_0 is a hermitian (or self-adjoint) operator: $L = L^\dagger$.

8.5 The Retarded Problem

8.5.1 General Solution of Retarded Problem

We now reduce 8.1 to a simpler problem. If this is an initial value problem, then $u(\mathbf{x}, t)$ is completely determined by equations 8.1, 8.2, pr:IVP2

pr:G1Id2

pr:G2Id2
22 Feb p3

pr:HermOp2

8.3, and 8.4. We look again at G_R which is the response of a system to a unit force:

$$\left[L_0 + \sigma(\mathbf{x}) \frac{\partial^2}{\partial t^2} \right] G_R(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \quad \text{for } \mathbf{x}, \mathbf{x}' \text{ in } R. \quad (8.8)$$

eq8tB

22 Feb p4

We also require the retarded Green's function to satisfy RBC and the initial condition $G_R = 0$ for $t < t'$. We now use the result from problem 4.2:

$$\begin{aligned} u(\mathbf{x}, t) = & \int_{\mathbf{x}' \in R} d\mathbf{x}' \int_0^t dt' G_R(\mathbf{x}, t; \mathbf{x}', t') \sigma(\mathbf{x}') f(\mathbf{x}', t') \\ & + \int_{\mathbf{x}' \in S} d\mathbf{x}' \tau(\mathbf{x}') \int_0^t dt' G_R(\mathbf{x}, t; \mathbf{x}', t') h(\mathbf{x}', t') \\ & + \int_{\mathbf{x}' \in R} d\mathbf{x}' \sigma(\mathbf{x}') \left[G_R(\mathbf{x}, t; \mathbf{x}', 0) u_1(\mathbf{x}') \right. \\ & \quad \left. - \frac{\partial}{\partial t'} G_R(\mathbf{x}, t; \mathbf{x}', t') u_0(\mathbf{x}') \right] \end{aligned} \quad (8.9)$$

eq8ugen

R Horn says
eval \hat{G}_R at \mathbf{x}
'=0

ask Baker
about limits

The first line gives the volume sources, the second gives the surface sources, and the third and fourth gives the contribution from the initial conditions. Since the defining equations for G_R are linear in volume, surface, and initial condition terms, we were able to write down the solution $u(\mathbf{x}, t)$ as a linear superposition of the G_R .

Note that we recover the initial value of $u(\mathbf{x}, t)$ in the limit $t \rightarrow t'$. This is true since in the above equation we can substitute

$$\lim_{t \rightarrow t'} G_R(\mathbf{x}, t; \mathbf{x}', t') = 0$$

and

$$\lim_{t \rightarrow t'} G_R(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sigma(\mathbf{x}')}.$$

pr:GR2

8.5.2 The Retarded Green's Function in N -Dim.

By using 8.9 we need only solve 8.8 to solve 8.1. In section 6.3 we found that G_R could be determined by using a Fourier Transform. Here we follow the same procedure generalized to N -dimensions. The Fourier transform of G_R in N -dimensions is

$$G_R(\mathbf{x}, t; \mathbf{x}', t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(\mathbf{x}, \mathbf{x}'; \omega)$$

where L is a line in the upper half plane parallel to the real axis (c.f., 22 Feb p5 section 6.4), since \tilde{G} is analytic in the upper half plane (which is due to the criterion $G_R = 0$ for $t < t'$).

Now the problem is simply to evaluate the Fourier Transform. By the same reasoning in section 6.3 the Fourier transform of G_R is identical to the Green's function for the steady state case:

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; \omega) = G(\mathbf{x}, \mathbf{x}; \lambda = \omega^2).$$

Recall that the Green's function for the steady state problem satisfies

$$[L_0 - \lambda\sigma]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') \quad \text{for } \mathbf{x}, \mathbf{x}' \in R, \text{RBC} \quad (8.10)$$

We have reduced the general problem in N dimensions (equation 8.1) to the steady state Green's function problem in N -dimensions. 22 Feb p6

8.5.3 Reduction to Eigenvalue Problem

pr:efp3

The eigenvalue problem (i.e., the homogeneous equation) in N -dimensions is

$$L_0 u_n(\mathbf{x}) = \lambda_n \sigma u_n(\mathbf{x}) \quad \mathbf{x} \in R, \text{RBC} \quad (8.11)$$

The λ_n 's are the eigen values of L_0 . Since L_0 is hermitian, the λ_n 's are real. The $u_n(\mathbf{x})$'s are the corresponding eigenfunctions of L_0 . We can also prove orthonormality (using the same method as in the single dimension case)

Lots of work

$$\int_R d\mathbf{x} \sum_{\alpha} u_n^{\alpha*}(\mathbf{x}) u_m^{\alpha}(\mathbf{x}) = 0 \text{ if } \lambda_n \neq \lambda_m.$$

This follows from the hermiticity of L_0 . Note that because we are now in N -dimensions, the degeneracy may now be infinite. 22 Feb p7

By using the same procedure as in chapter 4, we can write a solution of eq. 8.8 expanded in terms of a solution of 8.11:

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_n \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{\lambda_n - \lambda}.$$

Note that the sum would become an integral for a continuous spectrum.

The methods of chapter 4 also allow us to construct the δ -function 22 Feb p8

representation

$$\delta(\mathbf{x} - \mathbf{x}') = \sigma(\mathbf{x}') \sum_n u_n(\mathbf{x}) u_n^*(\mathbf{x}'),$$

which is also called the completeness relation. All we have left is to discuss the physical interpretation of G .

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8.6 Region R

24 Feb p3

8.6.1 Interior

pr:intprob2

In the interior problem the Green's function can be written as a discrete spectrum of eigenvalues. In the case of a discrete spectrum we have

$$G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2) = \sum_n \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{\lambda_n - \omega^2}$$

8.6.2 Exterior

pr:extprob2

For the exterior problem the sums become integrals and we have a continuous spectrum:

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \int_{\lambda_n} d\lambda_n \frac{\sum_{\alpha} u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{\lambda_n - \lambda}.$$

In this case we take $\lambda = \omega^2 + i\varepsilon$. G now has a branch cut for all real λ 's, which means that there will be two linearly independent solutions which correspond to whether we approach the real λ axis from above or below. We choose $\varepsilon > 0$ to correspond to the physical out going wave solution.

ask Baker

about omitted
material

24 Feb p4

8.7 The Method of Images

24 Feb p5

pr:MethIm1

We now present an alternative method for solving N -dimensional problem which is sometimes useful when the problem exhibits sufficient

symmetry. It is called the *Method of Images*. For simplicity we consider a one dimensional problem. Consider the G_R problem for periodic boundary conditions with constant coefficients.

$$\left(L_0 - \lambda\sigma \frac{\partial^2}{\partial t^2}\right) G_R = \delta(x - x')\delta(t - t') \quad 0 \leq x \leq l.$$

8.7.1 Eigenfunction Method

We have previously solved this problem by using an eigen function expansion solution (equation 6.31)

$$G_R = \sum_n \frac{u_n(x)u_n^*(x')}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t - t').$$

For this problem the eigen functions and eigen values are

$$u_n(x) = \left(\frac{1}{l}\right)^{1/2} e^{2\pi i n x / l} \quad n = 0, \pm 1, \pm 2, \dots$$

$$\lambda_n = \left(\frac{2\pi i n}{l}\right)^2 \quad n = 0, \pm 1, \pm 2, \dots$$

8.7.2 Method of Images

The method of images solution uses the uniqueness theorem. Put images over $-\infty$ to ∞ in region of length λ .

$$\Phi = \left(\frac{c}{2\tau}\right) \theta\left(t - t' - \frac{|x - x'|}{c}\right).$$

This is not periodic over 0 to l . Rather, it is over all space. Our G_R is

$$G_R = \frac{c}{2\tau} \sum_{n=-\infty}^{\infty} \theta\left(t - t' - \frac{|x - x' - nl|}{c}\right).$$

Notice that this solution satisfies

$$\left(L_0 + \sigma \frac{\partial^2}{\partial t^2}\right) G_R = \sum_{n=-\infty}^{\infty} \delta(x - x' - nl) \quad -\infty < x, x' < \infty.$$

However, we only care about $0 < x < l$

$$\left(L_0 + \sigma \frac{\partial^2}{\partial t^2}\right) G_R = \sum_{n=-\infty}^{\infty} \delta(x - x' - nl) \quad 0 < x, x' < l.$$

Since the other sources are outside the region of interest they do not affect this equation. Our Green's function is obviously periodic.

The relation between these solution forms is a Fourier series.

8.8 Summary

1. For the *exterior* problem, the region is outside the boundary and extends to the boundary. The *interior* problem, the boundary is inside the boundary and has finite extent. For the *all-space* problem, there is no boundary. The boundary conditions can be either elastic or periodic, or in the case that there is no boundary, the function must be regular at large and/or small values of its parameter.

2. The Green's identities for the N -dimensional case are

$$\langle S, L_0 u \rangle = - \int_S dS \hat{n} \cdot (S^* \tau(\mathbf{x}) \nabla u) + \int_R [S^* V u + (\nabla \cdot S^* \tau(\mathbf{x}) \nabla u)] d\mathbf{x},$$

$$\langle S, L_0 u \rangle - \langle L_0 S, u \rangle = \int_s dS \hat{n} \cdot [-S^* \tau(x) \nabla u + u \tau(x) \nabla S^*].$$

3. The solution for the N -dimensional problem in terms of the Green's function is

$$\begin{aligned} u(\mathbf{x}, t) = & \int_{\mathbf{x}' \in R} d\mathbf{x}' \int_0^t dt' G_R(\mathbf{x}, t; \mathbf{x}', t') \sigma(\mathbf{x}') f(\mathbf{x}', t') \\ & + \int_{\mathbf{x}' \in S} d\mathbf{x}' \tau(\mathbf{x}') \int_0^t dt' G_R(\mathbf{x}, t; \mathbf{x}', t') h(\mathbf{x}', t') \\ & + \int_{\mathbf{x}' \in R} d\mathbf{x}' \sigma(\mathbf{x}') \left[G_R(\mathbf{x}, t; \mathbf{x}', 0) u_1(\mathbf{x}') \right. \\ & \quad \left. - \frac{\partial}{\partial t'} G_R(\mathbf{x}, t; \mathbf{x}', t') u_0(\mathbf{x}') \right]. \end{aligned}$$

4. The method of images is applicable if the original problem exhibits enough symmetry. The method is to replace the original problem, which has a boundary limiting region of the solution, with a new problem in which the boundary is taken away and sources are placed in the region which was excluded by the boundary such that the solution will satisfy the boundary conditions of the original problem.

8.9 References

The method of images is covered in most electromagnetism books, for example [Jackson75, p54ff], [Griffiths81, p106ff]; a Green's function application is given in [Fetter80, p317]. The other material in this chapter is a generalization of the results from the previous chapters.

Chapter 9

Cylindrical Problems

Chapter Goals:

- Define the coordinates for cylindrical symmetry and obtain the appropriate δ -function.
- Write down the Green's function equation for the case of circular symmetry.
- Use a partial expansion for the Green's function to obtain the radial Green's function equation for the case of cylindrical symmetry.
- Find the Green's function for the case of a circular wedge and for a circular membrane.

29 Feb p1

9.1 Introduction

In the previous chapter we considered the Green's function equation

$$(L_0 - \lambda\sigma(\mathbf{x}))G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') \quad \text{for } \mathbf{x}, \mathbf{x}' \in R$$

where

$$L_0 = -\nabla \cdot (\tau(\mathbf{x})\nabla) + V(\mathbf{x})$$

subject to RBC, which are either for the elastic case

$$(\hat{n} \cdot \nabla + \kappa(S))G(\mathbf{x}, \mathbf{x}'; \lambda) = 0 \quad \text{for } \mathbf{x} \in S \quad (9.1)$$

or the periodic case. In this chapter we want to systematically solve this problem for 2-dimensional cases which exhibit cylindrical symmetry.

9.1.1 Coordinates

A point in space can be represented in cartesian coordinates as

$$\mathbf{x} = \hat{i}x + \hat{j}y.$$

pr:CartCoord1 Instead of the coordinate pair (x, y) we may choose polar coordinates (r, ϕ) . The transformation to cartesian coordinates is

$$x = r \cos \phi \quad y = r \sin \phi$$

pr:r1 while the transformation to polar coordinates is (for $\tan \phi$ defined on the interval $-\pi/2 < \phi < \pi/2$)

$$r = \sqrt{x^2 + y^2}$$

$$\phi = \begin{cases} \tan^{-1}(y/x) & \text{for } x > 0, y > 0 \\ \tan^{-1}(y/x) + \pi & \text{for } x < 0 \\ \tan^{-1}(y/x) + 2\pi & \text{for } x > 0, y < 0 \end{cases}$$

pr:jak1 A differential of area for polar coordinates is related by that for cartesian coordinates by a Jacobian (see Boas, p220):

$$\begin{aligned} dx dy &= dA \\ &= \left| J \left(\frac{x, y}{r, \phi} \right) \right| dr d\phi \\ &= \left| \left(\frac{\partial(x, y)}{\partial(r, \phi)} \right) \right| dr d\phi \\ &= \begin{vmatrix} (\partial x / \partial r) & (\partial x / \partial \phi) \\ (\partial y / \partial r) & (\partial y / \partial \phi) \end{vmatrix} dr d\phi \\ &= \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{vmatrix} dr d\phi \\ &= r dr d\phi \end{aligned}$$

By expanding dx and dy in terms of dr and $d\phi$, we can write the differential of arc length in polar coordinates (see Boas, p224)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\phi^2}.$$

The differential operator becomes (see Boas, p252,431)

pr:grad1

$$\begin{aligned} \text{gradient} \quad \nabla u &= \hat{r} \frac{\partial u}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial u}{\partial \phi}, \\ \text{divergence} \quad \nabla \cdot \mathbf{B} &= \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial}{\partial \phi} (B_\phi). \end{aligned}$$

Let $\mathbf{B} = \tau(\mathbf{x}) \nabla u$, then

$$\nabla \cdot (\tau(\mathbf{x}) \nabla u(\mathbf{x})) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \tau(\mathbf{x}) \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\tau(\mathbf{x}) \frac{\partial u}{\partial \phi} \right). \quad (9.2)$$

9.1.2 Delta Function

29 Feb p2

The N -dimensional δ -function is defined by the property

pr:DeltaFn2

$$f(\mathbf{x}) = \int d^2x f(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}').$$

In polar form we have (since $dx dy = r dr d\phi$)

$$f(\mathbf{x}) = f(r, \phi) = \int r' dr' d\phi' f(r', \phi') \delta(\mathbf{x} - \mathbf{x}').$$

By comparing this with

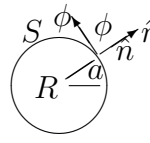
$$f(r, \phi) = \int dr' d\phi' f(r', \phi') \delta(r - r') \delta(\phi - \phi')$$

we identify that the delta function can be written in polar coordinates in the form

pr:delta1

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(r - r')}{r} \delta(\phi - \phi').$$

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Figure 9.1: The region R as a circle with radius a .

9.2 GF Problem for Cylindrical Sym.

The analysis in the previous chapters may be carried into cylindrical coordinates. For simplicity we consider cylindrical symmetry: $\tau(\mathbf{x}) = \tau(r)$, $\sigma(\mathbf{x}) = \sigma(r)$, and $V(\mathbf{x}) = V(r)$. Thus 9.2 becomes

$$\nabla(\tau(r)\nabla u(\mathbf{x})) = \frac{1}{r} \frac{\partial}{\partial r} \left(r\tau(r) \frac{\partial u}{\partial r} \right) + \frac{\tau(r)}{r^2} \frac{\partial^2 u}{\partial \phi^2}.$$

The equation for the Green's function

$$(L_0 - \lambda\sigma)G(r, \phi; r', \phi') = \frac{1}{r} \delta(r - r') \delta(\phi - \phi')$$

becomes (for $r, \phi \in R$)

$$\left[-\frac{1}{r} \frac{\partial}{\partial r} \left(r\tau(r) \frac{\partial}{\partial r} \right) - \frac{\tau(r)}{r^2} \frac{\partial^2}{\partial \phi^2} + V(r) - \lambda\sigma(r) \right] G = \frac{1}{r} \delta(r - r') \delta(\phi - \phi'). \quad (9.3)$$

Here R may be the interior or the exterior of a circle. It could also be a wedge of a circle, or an annulus, or anything else with circular symmetry.

For definiteness, take the region R to be the interior of a circle of radius a (see figure 9.1) and apply the elastic boundary condition 9.1. We now define the elasticity on the boundary S , $\kappa(S) = \kappa(\phi)$. We must further specify $\kappa(\phi) = \kappa$, a constant, since if $\kappa = \kappa(\phi)$, then we might not have cylindrical symmetry. Cylindrical symmetry implies $\hat{n} \cdot \nabla = \partial/\partial r$ so that the boundary condition

$$(\hat{n} \cdot \nabla + \kappa(S))G(r, \phi; r', \phi') = 0 \quad \text{for } r = a$$

is now

$$\left(\frac{\partial}{\partial r} + \kappa \right) G(r, \phi; r', \phi') = 0 \quad \text{for } r = a.$$

fig10a

We also need to have G periodic under $\phi \rightarrow \phi + 2\pi$. So

29 Feb p3

$$G(r, 0; r', \phi') = G(r, 2\pi; r', \phi')$$

and

$$\left. \frac{\partial G}{\partial \phi} \right|_{\phi=0} = \left. \frac{\partial G}{\partial \phi} \right|_{\phi=2\pi}.$$

We have now completely respecified the Green's function for the case of cylindrical symmetry.

9.3 Expansion in Terms of Eigenfunctions

Since the Green's function is periodic in ϕ and since ϕ only appears in the operator as $\partial^2/\partial\phi^2$, we use an eigenfunction expansion to separate out the ϕ -dependence. Thus we look for a complete set of eigenfunctions pr:efexp1 which solve

$$-\frac{\partial^2}{\partial\phi^2}u_m(\phi) = \mu_m u_m(\phi) \quad (9.4)$$

for $u_m(\phi)$ periodic. The solutions of this equation are

$$u_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{for } m = 0, \pm 1, \pm 2, \dots$$

and the eigenvalues are

$$\mu_m = m^2 \quad \text{for } m = 0, \pm 1, \pm 2, \dots$$

(Other types of regions would give different eigenvalues μ_m). Since pr:CompRel4 this set of eigenfunctions is complete it satisfies the expansion

$$\delta(\phi - \phi') = \sum_m u_m(\phi) u_m^*(\phi').$$

9.3.1 Partial Expansion

We now want to find $G_m(r, r'; \lambda)$ which satisfies the partial expansion (using the principle of superposition) pr:partExp1

$$G(r, \phi; r', \phi') = \sum_m u_m(\phi) G_m(r, r'; \lambda) u_m^*(\phi'). \quad (9.5)$$

We plug this and 9.4 into the partial differential equation 9.3:

$$\begin{aligned} \sum_m u_m(\phi) \left[-\frac{1}{r} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\mu_m \tau}{r^2} + V(r) - \lambda \sigma(r) \right] G_m u_m^*(\phi') \\ = \frac{1}{r} \delta(r - r') \sum_m u_m(\phi) u_m^*(\phi') \end{aligned}$$

ask Baker

where thi
comes from
29 Feb p4

pr:rlol

We now define the reduced linear operator

$$L_0^{\mu_m} \equiv r L_0 = -\frac{d}{dr} \left(r \tau(r) \frac{d}{dr} \right) + r \left[\frac{\mu_m \tau(r)}{r^2} + V(r) \right] \quad (9.6)$$

eq9rLo

so $G_m(r, r'; \lambda)$ must satisfy

$$(L_0^{\mu_m} - \lambda r \sigma(r)) G_m(r, r'; \lambda) = \delta(r - r'), \quad \text{for } 0 < r, r' < a$$

and the boundary condition

$$\left(\frac{\partial}{\partial r} + \kappa \right) G_m(r, r'; \lambda) = 0 \quad \text{for } r = a, 0 < r' < a.$$

Comments on the eigenvalues μ_m : The RBC will always lead to $\mu_n > 0$. If $\mu_m < 0$ then the term $\mu_m \tau(r)/r^2$ in 9.6 would act like an attractive sink and there would be no stable solution. Since $\mu_m > 0$, this term instead looks like a centrifugal barrier at the origin.

29 Feb p5

Note that the effective “tension” in this case is $r\tau(r)$, so $r = 0$ is a singular point. Thus we must impose regularity at $r = 0$: $|G(r = 0, r'; \lambda)| < \infty$.

9.3.2 Summary of GF for Cyl. Sym.

We have reduced the Green’s function for cylindrical symmetry to the 1-dimensional problem:

$$(L_0^{\mu_m} - \lambda r \sigma(r)) G_m(r, r'; \lambda) = \delta(r - r'), \quad \text{for } 0 < r, r' < a,$$

$$\left(\frac{\partial}{\partial r} + \kappa \right) G_m(r, r'; \lambda) = 0 \quad \text{for } r = a, 0 < r' < a,$$

$$|G(0, r'; \lambda)| < \infty.$$

9.4 Eigen Value Problem for L_0

To solve the reduced Green's function problem which we have just obtained, we must solve the reduced eigen value problem

pr:efp4

$$L_0^{\mu_m} u_n^{(m)}(r) = \lambda_n^{(m)} r \sigma(r) u_n^{(m)}(r) \quad \text{for } 0 < r < a,$$

$$\frac{du_n^{(m)}}{dr} + \kappa u_n^{(m)}(r) = 0 \quad \text{for } r = a,$$

$$|u_n^{(m)}(r)| < \infty \quad \text{at } r = 0.$$

In these equation $\lambda_n^{(m)}$ is the n th eigenvalue of the reduced operator $L_0^{(\mu_m)}$ and $u_n^{(m)}(r)$ is the n th eigenfunction of $L_0^{(\mu_m)}$. From the general theory of 1-dimensional problems (c.f., chapter 4) we know that

$$G_m(r, r'; \lambda) = \sum_n \frac{u_n^{(m)}(r) u_n^{*(m)}(r')}{\lambda_n^{(m)} - \lambda} \quad \text{for } m = 0, \pm 1, \pm 2, \dots$$

It follows that (using 9.5)

29 Feb p6

$$\begin{aligned} G(r, \phi, r', \phi'; \lambda) &= \sum_m u_m(\phi) \left(\sum_n \frac{u_n^{(m)}(r) u_n^{*(m)}(r')}{\lambda_n^{(m)} - \lambda} \right) u_m^*(\phi') \\ &= \sum_{n,m} \frac{u_n^{(m)}(r, \phi) u_n^{*(m)}(r', \phi')}{\lambda_n^{(m)} - \lambda} \end{aligned}$$

where $u_n^{(m)}(r, \phi) = u_m(\phi) u_n^{(m)}(r)$. Recall that G satisfies $(L_0 - \lambda \sigma)G = \delta(\mathbf{x} - \mathbf{x}')$. with RBC. Thus we can conclude

$$L_0 u_n^{(m)}(r, \phi) = \lambda_n^{(m)} \sigma(r) u_n^{(m)}(r, \phi) \quad \text{RBC.}$$

These $u_n^{(m)}(r, \phi)$ also satisfy a completeness relation

figure this part out

$$\begin{aligned} \sum_{m,n} u_n^{(m)}(r, \phi) u_n^{*(m)}(r', \phi') &= \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sigma(\mathbf{x})} \\ &= \frac{\delta(r - r') \delta(\phi - \phi')}{r' \sigma(r')} \end{aligned}$$

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The radial part of the Green's function, G_m , may also be constructed directly if solutions satisfying the homogeneous equation are known,

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where one of them also satisfies the $r = 0$ boundary condition and the other also satisfies the $r = a$ boundary condition. The method from chapter 3 (which is valid for 1-dimensional problems) gives

$$G_m(r, r'; \lambda) = -\frac{u_1(r_<)u_2(r_>)}{r\tau(r)W(u_1, u_2)}$$

$\tau(r)?$

where

$$(L_0^{\mu_m} - \lambda\sigma r)u_{1,2} = 0$$

$$|u_1| < \infty \quad \text{at } r = 0$$

$$\frac{\partial u_2}{\partial r} + \kappa u_2 = 0 \quad \text{for } r = a.$$

The effective mass density is $r\sigma(\tau)$, the effective tension is $r\tau(r)$, and the effective potential is $r(\mu_m\tau/r^2 + V(r))$.

9.5 Uses of the GF $G_m(r, r'; \lambda)$

2 Mar p3

9.5.1 Eigenfunction Problem

pr:efp5

Once $G_m(r, r'; \lambda)$ is known, the eigenvalues and normalized eigenfunctions can be found using the relation

$$G_m(r, r'; \lambda) \xrightarrow{\lambda \rightarrow \lambda_n^{(m)}} \frac{u_n^{(m)}(r)u_n^{(m)}(r')}{\lambda_n^{(m)} - \lambda}.$$

The eigen values come from the poles, the eigen functions come from the residues.

9.5.2 Normal Modes/Normal Frequencies

pr:NormMode4

In the general problem with no external forces the equation of motion is homogeneous

$$\left(L_0 + \sigma \frac{\partial^2}{\partial t^2}\right) u(\mathbf{x}, t) = 0 \quad + \text{RBC}.$$

this section is still rough We look for natural mode solutions:

$$u(\mathbf{x}, t) = e^{-i\omega_n^{(m)}t} u_n^{(m)}(r, \phi).$$

The natural frequencies are given by

$$\omega_n^{(m)} = \sqrt{\lambda_n^{(m)}}.$$

The eigen functions (natural modes) are (cf section 9.4)

$$u_n^{(m)}(r, \phi) = u_n^{(m)}(r) u_m(\phi).$$

The normal modes are

$$u_n^{(m)}(\mathbf{x}, t) = e^{-i\omega_n^{(m)}t} u_n^{(m)}(r, \phi).$$

The normalization of the factored eigenfunctions $u_n^{(m)}(r)$ and $u_m(\phi)$ is

$$\int_0^a dr r \sigma(r) u_n^{(m)}(r) u_{n'}^{*(m)}(r) = \delta_{n,n'} \quad \text{for } n = 1, 2, \dots$$

$$\int_0^{2\pi} d\phi u_m(\phi) u_{m'}^*(\phi) = \delta_{m,m'}.$$

The overall normalization of the (r, ϕ) eigen functions is

pr:normal3

$$\int_0^{2\pi} d\phi \int_0^a dr (r \sigma(r)) u_n^{(m)}(r, \phi) u_{n'}^{*(m')}(r, \phi) = \delta_{n,n'} \delta_{m,m'}$$

2 Mar p4

or

$$\begin{aligned} \int_0^{2\pi} \int_0^a r dr d\phi \sigma(r) u_n^{(m)}(r, \phi) u_{n'}^{*(m')}(r, \phi) &= \int_R d^3x \sigma(\mathbf{x}) u^*(\mathbf{x}) u(\mathbf{x}) \\ &= \delta_{n,n'} \delta_{m,m'}. \end{aligned}$$

9.5.3 The Steady State Problem

pr:sss2

This is the case of a periodic driving force:

$$\begin{aligned} \left(L_0 + \sigma \frac{\partial^2}{\partial t^2} \right) u(r, \phi, t) &= \sigma f(r, \phi) e^{-i\omega t} \\ \left(\frac{\partial}{\partial r} + \kappa \right) u(r, \phi, t) &= h(\phi) e^{-i\omega t} \end{aligned}$$

Note: As long as the normal mode solution has circular symmetry, we may perturb it with forces $f(r, \phi)$ and $h(\phi)$. It is not necessary to have circularly symmetric forces.

The solution is (using 9.5)

$$u(r, \phi) = \sum_m u_m(\phi) \times \left(\int_0^a r' \sigma(r') dr' G_m(r, r'; \lambda = \omega^2 + i\varepsilon) \int_0^{2\pi} d\phi' u_m^*(\phi') f(r', \phi') + G_m(r, a; \lambda = \omega^2 + i\varepsilon) \int_0^{2\pi} a d\phi' \tau(a) u_m^*(\phi') f(r', \phi') \right).$$

In this equation $\int_0^{2\pi} d\phi' u_m^*(\phi') f(r', \phi')$ is the m th Fourier coefficient of the interior force $f(r', \phi')$ and $\int_0^{2\pi} a d\phi' \tau(a) u_m^*(\phi') f(r', \phi')$ is the m th Fourier coefficient of the surface force $\tau(a)h(\phi')$.

9.5.4 Full Time Dependence

For the retarded Green's function we have

$$G_R(r, \phi, t; r', \phi', t') = \sum_m u_m(\phi) G_{mR}(r, t, r', t') u_m^*(\phi')$$

where

$$G_{mR}(r, t; r', t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_m(r, r'; \lambda = \omega^2)$$

$$G_m(r, r'; \lambda = \omega^2) = -\frac{1}{r\tau(r)} \frac{u_1(r_<)u_2(r_>)}{W(u_1, u_2)}.$$

Note: In the exterior case, the poles coalesce to a branch cut. All space has circular symmetry. All the normal limits ($\sum \rightarrow \int$, $\delta_{n,n'} \rightarrow \delta(n - n')$, etc.,) hold.

9.6 The Wedge Problem

We now consider the case of a wedge. The equations are similar for the internal and external region problems. We consider the internal region problem. The region R is now $0 < r < a$, $0 < \phi < \gamma$ and its boundary is formed by $\phi = 0$, $\phi - 2\pi$, and $r = a$.

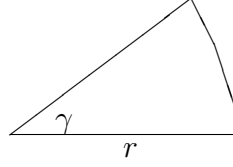


Figure 9.2: The wedge.

9.6.1 General Case

See the figure 9.2. The angular eigenfunction equation is again 9.4: *fig10.3*

$$-\frac{\partial^2}{\partial \phi^2} u_m(\phi) = \mu_m u_m(\phi) \quad \text{RBC.}$$

Note that the operator $\partial^2/\partial \phi^2$ is positive definite by Green's 1st identity. The angular eigenvalues are completely determined by the angular boundary conditions. For RBC it is always true the $\mu_m > 0$. This is physically important since if it were negative, the solutions to 9.4 would be real exponentials, which would not satisfy the case of periodic boundary conditions.

The boundary condition is now

$$(\hat{n} \cdot \nabla + \kappa)G = 0 \quad \mathbf{x} \in S.$$

This is satisfied if we choose $\kappa_1(r) = \kappa_1/r$, $\kappa_2(r) = \kappa_2/r$, and $\kappa_3(\phi) = \kappa_3$, with $\kappa_1, \kappa_2 \geq 0$. The boundary condition $(\hat{n} \cdot \nabla + \kappa)G = 0$ becomes

$$\begin{aligned} \left(-\frac{\partial}{\partial \phi} + \kappa_1\right)G &= 0 & \text{for } \phi = 0 \\ \left(\frac{\partial}{\partial \phi} + \kappa_2\right)G &= 0 & \text{for } \phi = \gamma \\ \left(\frac{\partial}{\partial r} + \kappa_3\right)G &= 0 & \text{for } r = a, 0 < \phi < \gamma. \end{aligned}$$

We now choose the $u_m(\phi)$ to satisfy the first two boundary conditions. The rest of the problem is the same, except that $L_0^{\mu_m}$ gives different μ_m eigenvalues.

9.6.2 Special Case: Fixed Sides

The case $\kappa_1 \rightarrow \infty$ and $\kappa_2 \rightarrow \infty$ corresponds to fixed sides. We thus have $G = 0$ for $\phi = 0$ and $\phi = \gamma$. So the u_m eigenvalues must satisfy

$$-\frac{\partial^2}{\partial \phi^2} u_m = \mu_m u_m$$

and

$$u_m = 0 \quad \text{for } \phi = 0, \gamma.$$

The solution to this problem is

$$u_m(\phi) = \sqrt{\frac{2}{\gamma}} \sin \frac{m\pi\phi}{\gamma}$$

with

$$\mu_m = \left(\frac{m\pi}{\gamma} \right)^2 \quad m = 1, 2, \dots$$

The case $m = 0$ is excluded because its eigenfunction is trivial. As $\gamma \rightarrow 2\pi$ we recover the full circle case.

ask Baker

why not $\gamma \rightarrow 2\pi$?

2 Mar p7

9.7 The Homogeneous Membrane

pr:membrane1

4 Mar p1

Recall the general Green's function problem for circular symmetry. By substituting the completeness relation for $u_m(\phi)$, our differential equation becomes

$$(L_0 - \lambda\sigma)G(x, x'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(r - r')}{r} \sum_m u_m(\phi) u_m^*(\phi')$$

where

$$L_0 u_m(\phi) = \frac{1}{r} L_0^{\mu_m} u_m(\phi),$$

$$L_0^{\mu_m} = -\frac{d}{dr} \left(r\tau(r) \frac{d}{dr} \right) + r \left(\frac{\mu_m \tau(r)}{r^2} + V(r) \right).$$

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We now consider the problem of a complete circle and a wedge.

We look at the case of a circular membrane or wedge with $V = 0$, $\sigma = \text{constant}$, $\tau = \text{constant}$. This corresponds to a homogeneous membrane. We separate the problem into radial and angular parts.

First we consider the radial part. To find $G_m(r, r'; \lambda)$, we want to solve the problem

$$\left[-\frac{d}{dr} \left(r \frac{d}{dr} \right) + \frac{\mu_m}{r} - \frac{\lambda r}{c^2} \right] G_m(r, r'; \lambda) = \frac{1}{\tau} \delta(r - r')$$

with $G = 0$ and $r = a$, which corresponds to fixed ends. This problem was solved in problem set 3:

$$\begin{aligned} G_m = & \frac{\pi}{2\tau} \frac{J_{\sqrt{\mu_m}}(r < \sqrt{\lambda/c^2})}{J_{\sqrt{\mu_m}}(a\sqrt{\lambda/c^2})} \left(J_{\sqrt{\mu_m}}(r > \sqrt{\lambda/c^2}) N_{\sqrt{\mu_m}}(a\sqrt{\lambda/c^2}) \right. \\ & \left. - J_{\sqrt{\mu_m}}(a\sqrt{\lambda/c^2}) N_{\sqrt{\mu_m}}(r > \sqrt{\lambda/c^2}) \right). \end{aligned} \quad (9.7)$$

Using 9.5, this provides an explicit solution of the full Green's function problem. Now we consider the angular part, where we have $\gamma = 2\pi$, so that $\sqrt{\mu_m} = \pm m$ which means the angular eigenfunctions are the same as for the circular membrane problem considered before:

$$u_m = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{for } \mu_m = m^2, m = 0, \pm 1, \pm 2, \dots$$

The total answer is thus a sum over both positive and negative m

$$G(r, \phi, r', \phi'; \lambda) = \sum_{m=-\infty}^{\infty} u_m(\phi) G_m(r, r'; \lambda) u_m^*(\phi').$$

We now redo this with $\kappa \rightarrow \infty$ and arbitrary γ . This implies that the eigen functions are the same as the wedge problem considered before

$$\begin{aligned} u_m(\phi) &= \sqrt{\frac{2}{\gamma}} \sin \left(\frac{m\pi\phi}{\gamma} \right), \\ \mu_m &= \left(\frac{m\pi}{\gamma} \right)^2. \end{aligned}$$

We now get $J_{m\pi/\gamma}(r\sqrt{\lambda/c^2})$ and $N_{m\pi/\gamma}(r\sqrt{\lambda/c^2})$. We also get the original expansion for G :

Show (new Bessel op

$$G(r, \phi; r', \phi'; \lambda) = \sum_{m=1}^{\infty} u_m(\phi) G_m(r, r'; \lambda) u_m^*(\phi')$$

9.7.1 The Radial Eigenvalues

pr:efp6

The poles of 9.7 occur when

$$J_{\sqrt{\mu_m}}(a\sqrt{\lambda/c^2}) = 0.$$

We denote the n th zero of $J_{\sqrt{\mu_m}}$ by $x_{\sqrt{\mu_m}n}$. This gives us

$$\lambda_{mn} = \left(\frac{x_{\sqrt{\mu_m}n} c}{a} \right)^2 \quad \text{for } n = 1, 2, \dots$$

where $J_{\sqrt{\mu_m}}(x_{\sqrt{\mu_m}n}) = 0$ is the n th root of the μ_m Bessel function. To find the normalized eigenfunctions, we look at the residues of

$$G_m \xrightarrow{\lambda \rightarrow \lambda_n} \frac{u_n^{(m)}(r) u_n^{(m)}(r')}{\lambda_n^{(m)} - \lambda}.$$

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We find

$$u_n^m(r) = \sqrt{\frac{2}{\sigma a^2}} \frac{J_{\sqrt{\mu_m}}(x_{\sqrt{\mu_m}n} \frac{r}{a})}{J'_{\sqrt{\mu_m}}(x_{\sqrt{\mu_m}n})}.$$

Thus the normalized eigen functions of the overall operator

$$L_0 u_n^{(m)}(r, \phi) = \sigma \lambda_n^{(m)} u_n^{(m)}(r, \phi)$$

are

$$u_n^{(m)}(r, \phi) = u_n^{(m)}(r) u_m(\phi)$$

where the form of $u_m(\phi)$ depends on whether we are considering a wedge or circular membrane.

9.7.2 The Physics

The normal mode frequencies are given by the radial eigenvalues

$$\omega_{m,n} = \sqrt{\lambda_n^{(m)}} = \frac{c}{a} x_{\sqrt{\mu},n}.$$

The eigen values increase in two ways: as n increases and as m increases. For small x (i.e., $x \ll 1$), $J_{\sqrt{\mu_m}} \sim (x)^{\sqrt{\mu_m}}$ which implies that for larger μ_m the rise is slower.

As m increases, μ_m increases, so the first root occurs at larger x . As we increase m , we also increase the number of angular nodes in $e^{im\phi}$ or $\sin(mn\phi/\gamma)$. This also increases the centrifugal potential. Thus $\omega_{m,2}$ is this true? increases with m . The more angular modes that are present, the more angular kinetic energy contributes to the potential barrier in the radial equation.

Now consider behavior with varying γ for a fixed m . μ_m increases as we decrease γ , so that $\omega_{m,n}$ increases. Thus the smaller the wedge, the larger the first frequency. The case $\gamma \rightarrow 0$ means the angular eigenfunctions oscillate very quickly and this angular energy gets thrown into the radial operator and adds to the centrifugal barrier.

9.8 Summary

1. Whereas cartesian coordinates measure the perpendicular distance from two lines, cylindrical coordinates measure the length of a line from some reference point in its angle from some reference line.
2. The δ -function for circular coordinates is

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(r - r')}{r} \delta(\phi - \phi').$$

3. The Green's function equation for circular coordinates is

$$\left[-\frac{1}{r} \frac{\partial}{\partial r} \left(r \tau(r) \frac{\partial}{\partial r} \right) - \frac{\tau(r)}{r^2} \frac{\partial^2 u}{\partial \phi^2} + V(r) - \lambda \sigma(r) \right] G = \delta(\mathbf{x} - \mathbf{x}').$$

4. The partial expansion of the Green's function for the circular problem is

$$G(r, \phi; r', \phi') = \sum_m u_m(\phi) G_m(r, r'; \lambda) u_m^*(\phi').$$

5. The radial Green's function for circular coordinates satisfies

$$(L_0^{\mu_m} - \lambda r \sigma(r)) G_m(r, r'; \lambda) = \delta(r - r'), \quad \text{for } 0 < r, r' < a,$$

where the reduced linear operator is

$$L_0^{\mu_m} \equiv r L_0 = -\frac{d}{dr} \left(r \tau(r) \frac{d}{dr} \right) + r \left[\frac{\mu_m \tau(r)}{r^2} + V(r) \right],$$

and the boundary condition

$$\left(\frac{\partial}{\partial r} + \kappa \right) G_m(r, r'; \lambda) = 0 \quad \text{for } r = a, 0 < r' < a.$$

9.9 Reference

The material in this chapter can be also found in various parts of [Fetter80] and [Stakgold67].

The preferred special functions reference for physicists seems to be [Jackson75].

Chapter 10

Heat Conduction

Chapter Goals:

- Derive the conservation law and boundary conditions appropriate for heat conduction.
- Construct the heat equation and the Green's function equation for heat conduction.
- Solve the heat equation and interpret the solution.

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10.1 Introduction

pr:heat1

We now turn to the problem of heat conduction.¹ The following physical parameters will be used: mass density ρ , specific heat per unit mass c_p , temperature T , and energy E . Again we consider a region R with boundary S and outward normal \hat{n} .

10.1.1 Conservation of Energy

The specific heat, c_p , gives the additional amount of thermal energy which is stored in a unit of mass of a particular material when its temperature is raised by one unit: $\Delta E = c_p \Delta T$. Thus the total energy can be expressed as

$$E_{total} = E_0 + \int_R d^3x \rho c_p T.$$

¹The corresponding material in FW begins on page 408

Differentiating with respect to time gives

$$\frac{dE}{dt} = \int_R d^3x \rho c_p \left(\frac{\partial T}{\partial t} \right).$$

There are two types of energy flow: from across the boundary S and from sources/sinks in R .

1. Energy flow into R across S . This gives

$$\left(\frac{dE}{dt} \right)_{\text{boundary}} = - \int \hat{n} \cdot \mathbf{j}_n dS = - \int_R d\mathbf{x} \nabla \cdot \mathbf{j}_n$$

pr:heatcur1

where the heat current is defined

$$\mathbf{j}_n = -k_T \nabla T.$$

k_T is the thermal conductivity. Note that since ∇T points toward the hot regions, the minus sign in the equation defining heat flow indicates that heat flows from hot to cold regions.

2. Energy production in R due to sources or sinks,

$$\left(\frac{dE}{dt} \right)_{\text{sources}} = - \int_R d^3x \rho \dot{q}$$

where \dot{q} is the rate of energy production per unit mass by sources inside S .

Thus the total energy is given by

$$\begin{aligned} \int_R d^3x \rho c_p \frac{\partial T}{\partial t} &= \frac{dE}{dt} \\ &= \left(\frac{dE}{dt} \right)_{\text{boundary}} + \left(\frac{dE}{dt} \right)_{\text{sources}} \\ &= \int_R d^3x (\rho \dot{q} - \nabla \cdot \mathbf{j}_n). \end{aligned}$$

By taking an arbitrary volume, we get the relation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k_T \nabla T) + \rho \dot{q}. \quad (10.1)$$

10.1.2 Boundary Conditions

There are three types of boundary conditions which we will encounter:

1. T given on S . This is the case of a region surrounded by a heat bath.
2. $\hat{n} \cdot \nabla T$ for $x \in S$ given. This means that the heat current normal to the boundary, $\hat{n} \cdot \mathbf{j}_n$, is specified. In particular, if the boundary is insulated, then $\hat{n} \cdot \nabla T = 0$.
3. $-k_T(\mathbf{x})\hat{n} \cdot \nabla T = \alpha(T - T_{\text{external}})$ for $x \in S$.

In the first case the temperature is specified on the boundary. In the second case the temperature flux is specified on the boundary. The third case is a radiation condition, which is a generalization of the first two cases. The limiting value of α give

$$\begin{aligned} \alpha \gg 1 &\implies T \approx T_{\text{external}} \rightarrow \#1 \\ \alpha \ll 1 &\implies \hat{n} \cdot \nabla T \approx 0 \rightarrow \#2 \text{ with insulated boundary.} \end{aligned}$$

We now rewrite the general boundary condition (3) as

$$[\hat{n} \cdot \nabla T + \theta(S)]T(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in S \quad (10.2)$$

where $\theta(S) = \alpha/k_T(S)$ and $h(S, t) = (\alpha/k_T(S))T_{\text{external}}$. In the limit $\theta \gg 1$, T is given. In this case we recover boundary condition #1. The radiation is essentially perfect, which says that the temperature of the surface is equal to the temperature of the environment, which corresponds to $\alpha \rightarrow \infty$. In the other limit, for $\theta \ll 1$, $\hat{n} \cdot \nabla T$ is given. Thus we recover boundary condition # 2 which corresponds to $\alpha \rightarrow 0$.

By comparing the general boundary conditions for the heat equation with the general N -dimensional elastic boundary condition,

$$[\hat{n} \cdot \nabla + \kappa(\mathbf{x})]u(\mathbf{x}, t) = h(\mathbf{x}, t)$$

we identify $u(\mathbf{x}, t) \rightarrow T(\mathbf{x}, t)$ and $\kappa(\mathbf{x}) \rightarrow \theta(\mathbf{x})$.

10.2 The Standard form of the Heat Eq.

10.2.1 Correspondence with the Wave Equation

We can make the conservation of energy equation 10.1 look more familiar by writing it in our standard differential equation form

$$\left(L_0 + \rho c_p \frac{\partial}{\partial t} \right) T = \rho \dot{q}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in G \quad (10.3)$$

where the linear operator is

$$L_0 = -\nabla \cdot (k_T(\mathbf{x}) \nabla).$$

The correspondence with the wave equation is as follows:

Wave Equation	Heat Equation
$\tau(\mathbf{x})$	$k_T(\mathbf{x})$
$\sigma(\mathbf{x})$	$\rho(\mathbf{x})c_p$
$\sigma(\mathbf{x})f(\mathbf{x}, t)$	$\rho(\mathbf{x})\dot{q}(\mathbf{x}, t)$
$V(\mathbf{x})$	no potential

For the initial condition, we only need $T(\mathbf{x}, 0)$ to fully specify the solution for all time.

10.2.2 Green's Function Problem

We know that because equation 10.3 is linear, it is sufficient to consider only the Green's function problem (which is related to the above problem by $p\dot{q}(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$ and $h(S, t) = 0$):

$$\left(L_0 + \rho c_p \frac{\partial}{\partial t} \right) G(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t'),$$

$$[\hat{n} \cdot \nabla + \theta(S)]G(\mathbf{x}, t; \mathbf{x}', t') = 0 \quad \text{for } \mathbf{x} \in S,$$

$$G(\mathbf{x}, t; \mathbf{x}', t') = 0 \quad \text{for } t < t'.$$

We lose symmetry in time since only the first time derivative appears. We evaluate the retarded Green's functions by applying the standard Fourier transform technique from chapter 6:

$$G(\mathbf{x}, t; \mathbf{x}', t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(\mathbf{x}, \mathbf{x}'; \omega).$$

We know by $G = 0$ for $t < t'$ that \tilde{G} is analytic in the $\text{Im } \omega > 0$ plane. Thus we take L to be a line parallel to the real ω -axis in the upper half plane. The Fourier Transform of the Green's function is the solution of the problem

$$\begin{aligned} (L_0 - \rho c_p i\omega) \tilde{G}(\mathbf{x}, \mathbf{x}'; \omega) &= \delta(\mathbf{x} - \mathbf{x}'), \\ (\hat{n} \cdot \nabla + \theta(\mathbf{x})) \tilde{G}(\mathbf{x}, \mathbf{x}'; \omega) &= 0 \quad \text{for } \mathbf{x} \in S, \end{aligned}$$

which is obtained by Fourier transforming the above Green's function problem.

10.2.3 Laplace Transform

pr:LapTrans1

We note that this problem is identical to the forced oscillation Green's function problem with the substitutions $\sigma \rightarrow \rho c_p$ and $\tau \rightarrow k_T$. Thus we identify

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; \omega) = G(\mathbf{x}, \mathbf{x}'; \lambda = i\omega).$$

The single time derivative causes the eigenvalues to be $\lambda = i\omega$. To evaluate this problem we thus make the substitution $s = -i\omega$. This substitution results in the *Laplace Transformation*. Under this trans- 7 Mar p4 formation the Green's function in transform space is related by

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; \omega = is) = G(\mathbf{x}, \mathbf{x}'; \lambda = -s).$$

\tilde{G} is now analytic in the right hand side plane: $\text{Re}(s) > 0$. This variable substitution is depicted in figure 10.1. The transformed contour is labeled L' . The Laplace transform of the Green's function satisfies *fig10a* the relation

$$G(\mathbf{x}, t; \mathbf{x}', t') = \frac{i}{2\pi} \int_{L' \downarrow} ds \tilde{G}(\mathbf{x}, \mathbf{x}'; \lambda = -s') e^{s(t-t')}$$

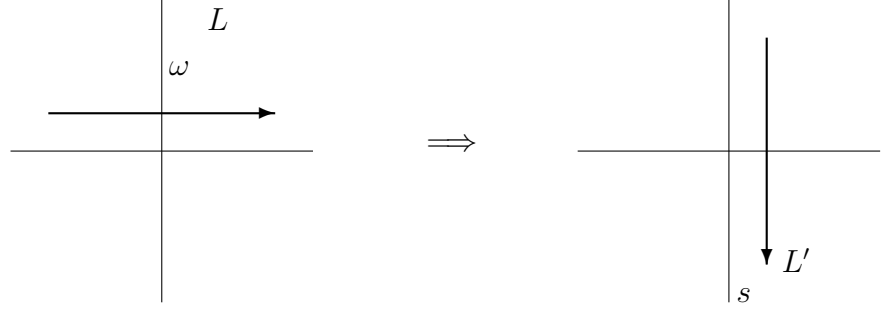


Figure 10.1: Rotation of contour in complex plane.

or, by changing the direction of the path, we have

$$G(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{2\pi i} \int_{L' \uparrow} ds \tilde{G}(\mathbf{x}, \mathbf{x}'; \lambda = s') e^{s(t-t')}. \quad (10.4)$$

In the following we will denote $L' \uparrow$ as L .

The inversion formula

$$\tilde{G}(\omega) = \int_0^\infty d\tau e^{i\omega\tau} G(\mathbf{x}, \mathbf{x}', \tau = t - t')$$

is also rotated to become

$$\tilde{G}(s) = \int_0^\infty d\tau e^{-s\tau} G(\mathbf{x}, \mathbf{x}'; \tau).$$

$\tilde{G}(s)$ is analytic for all $\text{Re}(s) > 0$. Note that the retarded condition allows us start the lower limit at $\tau = 0$ rather than $\tau = -\infty$.

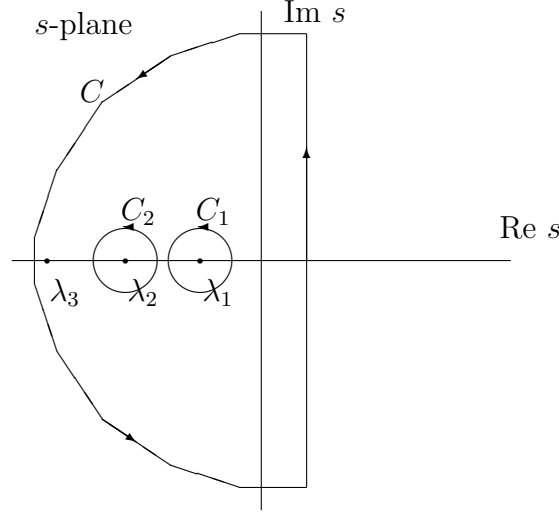
10.2.4 Eigen Function Expansions

We now solve the Green's function by writing it as a bilinear sum of eigenfunctions:

$$G(x, x'; \lambda) = \sum_n \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{\lambda_n - \lambda}. \quad (10.5)$$

The eigenfunctions $u_n(\mathbf{x})$ solve the problem

$$L_0 u_n(\mathbf{x}) = \lambda_n \rho c_p u_n(x) \quad \text{for } \mathbf{x} \in R$$

Figure 10.2: Contour closed in left half s -plane.

where $L_0 = -\nabla \cdot (k_T(\mathbf{x})\nabla)$ with the elastic boundary condition

$$(\hat{n} \cdot \nabla + \theta(s))u_n = 0 \quad \text{for } \mathbf{x} \in S.$$

Because of the identification

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; s) = G(\mathbf{x}, \mathbf{x}'; \lambda = -s)$$

we can substitute 10.5 into the transform integral 10.4 to get

$$\begin{aligned} G(\mathbf{x}, t; \mathbf{x}', t') &= \int_L \frac{ds}{2\pi i} \sum_n \frac{u_n(\mathbf{x})u_n^*(\mathbf{x}')}{\lambda_n + s} e^{s(t-t')} \\ &= \sum_n u_n(\mathbf{x})u_n^*(\mathbf{x}') \frac{1}{2\pi i} \oint_L \frac{ds}{\lambda_n + s} e^{s(t-t')}. \end{aligned}$$

This vanishes for $t < t'$. Close the contour in the left half s -plane for $t - t' > 0$, as shown in figure 10.2. This integral consists of contributions from the residues of the poles at $-\lambda_n$, where $n = 1, 2, \dots$. So

$$\frac{1}{2\pi i} \oint_{C_n} \frac{ds}{\lambda_n + s} e^{s(t-t')} = e^{-\lambda_n(t-t')}.$$

Thus

$$G(\mathbf{x}, t; \mathbf{x}', t) = \sum_n u_n(\mathbf{x})u_n^*(\mathbf{x}') e^{-\lambda_n(t-t')}. \quad (10.6)$$

We now consider the two limiting cases for t .

Suppose that $t \rightarrow t'$. Then 10.6 becomes

$$G \xrightarrow{t \rightarrow t'} \sum_n u_n(\mathbf{x}) u_n^*(\mathbf{x}') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\rho c_p}.$$

7 Mar p6

Thus we see that another interpretation of G is as the solution of an initial value problem with the initial temperature

$$T(\mathbf{x}, 0) = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\rho c_p}$$

why is this?

and no forcing term.

Now suppose we have the other case, $t - t' \gg 1$. We know $\lambda_n > 0$ for all n since L_0 is positive definite (physically, entropy requires $k > 0$ so that heat flows from hot to cold). Thus the dominant term is the one for the lowest eigenvalue:

$$G \sim u_1(\mathbf{x}) u_1^*(\mathbf{x}') e^{-\lambda_1(t-t')} \quad (t - t') \gg 1.$$

In particular, this formula is valid when $(t - t') > 1/\lambda_2$. We may thus interpret $1/\lambda_n = \tau_n$ as the lifetime of these states. After $(t - t') \gg \tau_N$, all contributions to G from eigen values with $n \geq N$ are exponentially small.

This is the physical meaning for the eigen values. The reason that the lowest eigen function contribution is the only one that contributes for $t - t' \gg 1$ is because for higher N there are more nodes in the eigenfunction, so it has a larger spatial second derivative. This means (using the heat equation) that the time derivative of temperature is large, so the temperature is able to equilibrate quickly. This smoothing or diffusing process is due to the term with a first derivative in time, which gives the non-reversible nature of the problem.

10.3 Explicit One Dimensional Calculation

9 Mar p1

We now consider the heat equation in one dimension.

10.3.1 Application of Transform Method

Recall that the 1-dimensional Green's function for the free space wave equation is defined by

$$(L_0 - \sigma\lambda)G = \delta(x - x') \quad \text{for } -\infty < x < \infty.$$

We found that the solution for this wave equation is

$$G(x, x'; \lambda) = \frac{1}{2\sqrt{\lambda}} \frac{e^{i\sqrt{\lambda}|x-x'|/c}}{\sigma c}.$$

Transferring from the wave equation to the heat equation as discussed above, we substitute $\tau \rightarrow k_T$, $\sigma \rightarrow \rho$, $c = \sqrt{\tau/\sigma} \rightarrow \sqrt{\kappa}$ where $\kappa = K_T/\rho c_p$ is the thermal diffusivity, and $\sqrt{\lambda} \rightarrow i\sqrt{s}$ which means $\text{Im } \lambda > 0$ becomes $\text{Re } s > 0$. The substitutions yield

$$\tilde{G}(x, x'; s) = \left(\frac{1}{2\sqrt{s}} \rho c_p \sqrt{\kappa} \right) e^{-\sqrt{s} \left(\frac{|x-x'|}{\sqrt{\kappa}} \right)}$$

or

$$\rho c_p \tilde{G}(x, x'; s) = \frac{1}{2\sqrt{\kappa s}} e^{-\sqrt{s/\kappa}|x-x'|}.$$

We see that $\sqrt{\kappa}$ plays the role of a velocity. Now invert the transform to obtain the free space Green's function for the heat equation:

$$\begin{aligned} \rho c_p G(x, t; x', t') &= \int_L \frac{ds}{2\pi i} e^{s(t-t')} \rho c_p \tilde{G}(x, x'; s) \\ &= \int_L \frac{ds}{2\pi i} \frac{e^{s(t-t') - \sqrt{s/\kappa}|x-x'|}}{2\sqrt{s\kappa}}. \end{aligned}$$

10.3.2 Solution of the Transform Integral

Our result has a branch on \sqrt{s} . We parameterize the s -plane:

$$s = |s|e^{i\theta} \quad \text{for } -\pi < \theta < \pi$$

This gives us $\text{Re } \sqrt{s} = |s|^{1/2} \cos(\theta/2) > 0$. We choose the contour of integration based on t .

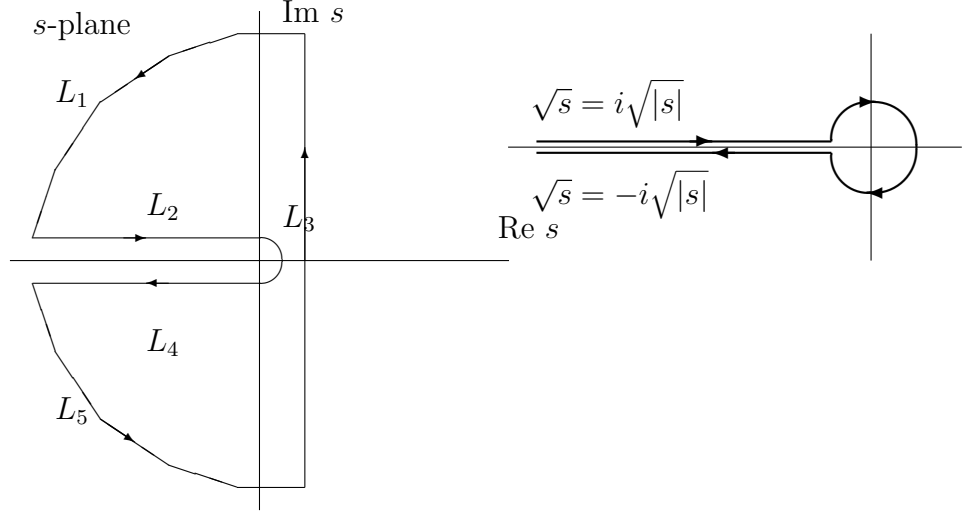


Figure 10.3: A contour with Branch cut.

For $t < t'$ we have the condition $G = 0$. Thus we close the contour in the right half plane so that

$$\exp \left[s(t - t') - \sqrt{s}|x - x'|/\sqrt{\kappa} \right] \xrightarrow{s \rightarrow \infty} 0$$

since both terms are increasingly negative. Since the contour encloses no poles, we recover $G = 0$ as required.

For $t - t' > 0$, close contour in the left half plane. See figure 10.3. We know by Cauchy's theorem that the integral around the closed contour $L + L_1 + L_2 + L_3 + L_4 + L_5$ vanishes. We perform the usual Branch cut evaluation, by treating the different segments separately. For L_3 it is convenient to use the parameterization $s = \varepsilon e^{i\theta}$ for $-\pi < \theta < \pi$ as shown in figure 10.3. In this case the integral becomes

$$\frac{1}{2\pi} \frac{1}{2\sqrt{\kappa}} \int_{-\pi}^{\pi} d\theta |\sqrt{\varepsilon}| [1 + \mathcal{O}(\varepsilon(t - t')) + \mathcal{O}(\varepsilon^{1/2}|x - x'|/\sqrt{\kappa})] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

In this equation we assert that it is permissible to take the limit $\varepsilon \rightarrow 0$ before the other quantities are taken arbitrarily large.

For the contour L_2 above the branch cut we have $\sqrt{s} = i\sqrt{|s|}$, and for the contour L_4 below the branch cut we have $\sqrt{s} = -i\sqrt{|s|}$

Combining the integrals for these two cases gives

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{-\varepsilon} \frac{ds}{2\pi} \frac{e^{s(t-t')}}{2\sqrt{\kappa}} \frac{2 \cos \sqrt{|s|/\kappa} |x - x'|}{\sqrt{|s|}}.$$

For L_1 and L_5 the integral vanishes. By letting $s = Re^{i\theta}$ where $-\pi < \theta < \pi$ we have

$$\left| \frac{\exp[-\sqrt{s}|x - x'|/\sqrt{\kappa}]}{\sqrt{s}} \right| \leq \frac{\exp\left[-\frac{|x-x'|}{\sqrt{\kappa}} R^2 \cos \frac{1}{2}\phi\right]}{\sqrt{R}} \xrightarrow{R \rightarrow \infty} 0.$$

Our final result is

$$\rho c_p G(x, t; x', t') = \frac{1}{2\pi\sqrt{\kappa}} \int_0^\infty \frac{ds}{\sqrt{s}} e^{-s(t-t')} \cos \sqrt{s/\kappa} (x - x').$$

Substituting $s = u^2$ gives

$$\rho c_p G(x, t; x', t') = \int_0^\infty \frac{2u du}{2\pi u \sqrt{\kappa}} \cos \frac{u}{\sqrt{\kappa}} |x - x'| e^{-u^2(t-t')}$$

or

$$\rho c_p G(x, t; x', t') = \frac{1}{\sqrt{\kappa}} I(t - t', |x - x'|/\sqrt{\kappa})$$

where (since the integrand is even)

$$I(t, y) = \frac{1}{2} \int_{-\infty}^\infty \frac{du}{\pi} e^{-u^2 t + iuy}.$$

This can be made into a simple Gaussian by completing the square: pr:gaus1

$$I(t, y) = e^{-y^2/(4t)} \int_{-\infty}^\infty \frac{du}{2\pi} e^{\pm(u - \frac{iy}{2t})^2}.$$

By shifting $u \rightarrow u + iy/2t$, the result is

9 Mar p6

$$I(t, y) = \frac{e^{-y^2/(4t)}}{\sqrt{4\pi t}}.$$

The free space Green function in 1-dimension is thus

$$\rho c_p G(x, t; x', t') = \left(\frac{1}{\sqrt{4\pi\kappa|t - t'|}} \right) e^{\frac{-(x-x')^2}{4\kappa(t-t')}}. \quad (10.7)$$

10.3.3 The Physics of the Fundamental Solution

This solution corresponds to a pure initial value problem where, if $x' = t' = 0$, we have

$$\rho c_p G(x, t) = \frac{e^{-x^2/4\kappa t}}{\sqrt{4\pi\kappa t}}.$$

At the initial time we have

$$\rho c_p G \xrightarrow{t \rightarrow 0} \delta(x - x') = \delta(x).$$

1. For $x^2 > 4\kappa t$, the amplitude is very small. Since G is small for $x \geq \sqrt{4\kappa t}$, diffusion proceeds at rate proportional to \sqrt{t} , not t as in wave equation. The average propagation is proportional to $t^{1/2}$. This is indicative of a statistical process (Random walk). It is non-dynamical in that it does not come from Newton's laws. Rather it comes from the dissipative-conduction nature of thermodynamics.
2. For any $t > 0$ we have a non-zero effect for all space. This corresponds to propagation with infinite velocity. Again, this indicates the non-dynamical nature of the problem. This is quite different from the case of wave propagation, where an event at the origin does not affect the position x until time x/c .
3. Another non-dynamical aspect of this problem is that it smoothes the singularity in the initial distribution, whereas the wave equation propagates all singularities in the initial distribution forward in time.
4. κ is a fundamental parameter whose role for the heat equation is analogous to the role of c for the wave equation. It determines the rate of diffusion. $\kappa (= k_T/\rho c_p)$ has the dimensions of (distance)²/time, whereas c has the dimensions of distance/time.

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11 Mar p1

10.3.4 Solution of the General IVP

We now use the Green's function to solve the initial value problem:

$$\left(-k_T \frac{\partial^2}{\partial x^2} + \rho c_p \frac{d}{dt} \right) T(x, t) = 0 \quad \text{for } -\infty < x, x', \infty$$

11 Mar p2

$$T(x, 0) = T_0(x)$$

$$T \rightarrow 0 \quad \text{for } |x| \rightarrow \infty$$

The method of the solution is to use superposition and 10.8:

$$\begin{aligned}
 T(x, t) &= \int_{-\infty}^{\infty} dx' T_0(x') \rho c_p G(x', 0; x, t) \\
 &= \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} dx' e^{-(x-x')^2/(4\kappa t)} T_0(x') \quad (10.8)
 \end{aligned}$$

10.3.5 Special Cases

Initial δ -function

Suppose $T_0(x) = \delta(x - x')$. Then we have $T(x, t) = \rho c_p G(x'', 0; x, t)$. Thus we see that G is the solution to the IVP with the δ -function as the initial condition and no forcing term.

Initial Gaussian Function

pr:gaus2

We now consider the special case of an initial Gaussian temperature distribution. Let $T_0(x) = (a/\pi)^{1/2} e^{-ax^2}$. The width of the initial distribution is $(\Delta x)_0 = 1/\sqrt{a}$. Plugging this form of $T_0(x)$ into 10.8 gives

$$\begin{aligned}
 T(x, t) &= \frac{1}{\pi\sqrt{4\kappa t a}} \int_{-\infty}^{\infty} dx' e^{-(x-x')^2/(4\kappa t) - ax'^2} \\
 T(x, t) &= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{(1/a) + 4\kappa t}} e^{-\frac{x^2}{(1/a) + 4\kappa t}} \\
 &= \frac{1}{\sqrt{\pi}} \left(\frac{1}{\Delta x} \right) e^{-(x/\Delta x)^2}
 \end{aligned}$$

where $\Delta x = \sqrt{(\Delta x_0)^2 + 4\kappa t}$. The packet is spreading as $(\Delta x)^2 = 4\kappa t + (\Delta x_0)^2$. Again, $\Delta x \sim t^{1/2}$ like a random walk, (again non-dynamical). Suppose $t \gg \tau \equiv (\Delta x_0)^2/4\kappa$. This is the simplest quantity with dimensions of time, so τ is the characteristic time of the system. We rewrite $\Delta x = (\Delta x_0)\sqrt{1 + t/\tau}$. Thus for $t \gg \tau$,

$$\Delta x \sim \sqrt{t/\tau} (\Delta x_0).$$

11 Mar p3

$\tau = (\Delta x_0)^2/4\kappa$ is a fundamental unit of time in the problem. Since ask Baker the region is infinite, there does not exist any characteristic distance about omitted for the problem.

11 Mar p4

11 Mar p5

10.4 Summary

1. Conservation of energy for heat conduction is given by the equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k_T \nabla T) + \rho \dot{q},$$

where ρ is the mass density, c_p is the specific heat, T is the temperature, k_T is the thermal conductivity, and \dot{q} is the rate of energy production per unit mass by sources inside the region.

2. The general boundary condition for heat conduction is

$$[\hat{n} \cdot \nabla T + \theta(S)]T(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in S.$$

3. The heat equation is

$$\left(L_0 + \rho c_p \frac{\partial}{\partial t} \right) T = \rho \dot{q}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in G,$$

where the linear operator is

$$L_0 = -\nabla \cdot (k_T(\mathbf{x}) \nabla).$$

4. The Green's function equation for the heat conduction problem is

$$\left(L_0 + \rho c_p \frac{\partial}{\partial t} \right) G(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}') \delta(t - t').$$

5. The solution of the heat equation for the initial value problem in one dimension is

$$T(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} dx' e^{-(x-x')^2/(4\kappa t)} T_0(x'),$$

which is a weighted integration over point sources which individually diffuse with a gaussian shape.

10.5 References

A similar treatment (though more thorough) is given in [Stakgold67b, p194ff]. See also [Fetter80, p406ff].

The definitive reference on heat conduction is [Carslaw86].

Chapter 11

Spherical Symmetry

Chapter Goals:

- Derive the form of the linear operator in spherical coordinates.
- Show that the angular part of the linear operator $L_{\theta\phi}$ is hermitian.
- Write the eigenvalue equations for Y_l^m .
- Write the partial wave expansion for the Green's function.
- Find the Green's function for the free space problem.

28 Mar p1 (17)

Our object of study is the Green's function for the problem

$$[L_0 - \lambda\sigma(\mathbf{x})]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') \quad (11.1)$$

with the regular boundary condition (RBC)

eq11.1

$$[\hat{n} \cdot \nabla + K(S)]G(\mathbf{x}, \mathbf{x}'; \lambda) = 0$$

for \mathbf{x}' in a region R and \mathbf{x} in the regions boundary S . The term \mathbf{x} is a field point, and \mathbf{x}' is a source point. The unit vector \hat{n} is the outward normal of the surface S . The operator L_0 is defined by the equation

$$L_0 = -\nabla \cdot (\tau(\mathbf{x})\nabla) + V(\mathbf{x}).$$

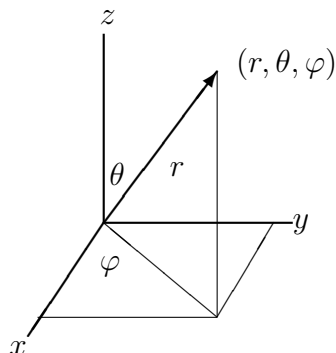


Figure 11.1: Spherical Coordinates.

We have solved this problem for the one and two dimensional cases in which there was a certain degree of symmetry.

11.1 Spherical Coordinates

28 Mar p2

pr:spher1

We now treat the problem in three dimensions. For this we use spherical coordinates (since we will later assume angular independence). A point in spherical coordinates is denoted (r, θ, φ) , where the range of each variable is

$$\begin{aligned} 0 &\leq r < \infty, \\ 0 &< \theta \leq \pi, \\ 0 &\leq \varphi < 2\pi. \end{aligned}$$

We use the following transformation of coordinate systems:

$$\begin{aligned} z &= r \cos \theta, \\ x &= r \sin \theta \cos \varphi, \\ y &= r \sin \theta \sin \varphi. \end{aligned}$$

fig11.1

This relationship is illustrated in figure 11.1 For an arbitrary volume

element we have

$$\begin{aligned} d^3x &= (dr)(r d\theta)(r \sin \theta d\varphi) \\ &= r^2 d\Omega dr \end{aligned}$$

where Ω is the solid angle, and an infinitesimal of solid angle is $d\Omega = \sin \theta d\theta d\varphi$.

We further define the delta function

$$\begin{aligned} f(r, \theta, \varphi) &= f(\mathbf{x}) \\ &= \int d^3x' f(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \\ &= \int dr' r'^2 \sin \theta' d\theta' d\varphi' f(r', \theta', \varphi') \delta(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

From this we can extract the form of the δ -function for spherical coordinates:

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x}') &= \frac{1}{r^2 \sin \theta} \delta(r - r') \delta(\theta - \theta') \delta(\varphi - \varphi') \\ &= \frac{\delta(r - r')}{r^2} \delta(\Omega - \Omega') \end{aligned}$$

where the solid angle δ -function is

$$\delta(\Omega - \Omega') = \frac{\delta(\theta - \theta') \delta(\varphi - \varphi')}{\sin \theta}.$$

We want to rewrite equation 11.1 in spherical coordinates. First we define the gradient

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r \sin \theta} \frac{\partial}{\partial \varphi}.$$

See [Boas] for derivations of identities involving ∇ . The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} A_\varphi.$$

When we apply this to the case

$$\mathbf{A} = \tau(\mathbf{x}) \nabla.$$

28 Mar p3

pr:delta2

pr:grad2

87' notes have
Gauss' law,
p18

the result is

$$\begin{aligned}\nabla \cdot (\tau \nabla) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \tau \frac{\partial}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\tau}{r} \frac{\partial}{\partial \theta} \right) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(\frac{\tau}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)\end{aligned}\quad (11.2)$$

eq11.2

28 Mar p4

where $\tau = \tau(\mathbf{x}) = \tau(r, \theta, \varphi)$.

Now we can write L_0 . We assume that τ , σ , and V are spherically symmetric, i.e., they are only a function of r : $\tau(\mathbf{x}) = \tau(r)$, $\sigma(\mathbf{x}) = \sigma(r)$, $V(\mathbf{x}) = V(r)$. In this case the linear operator is

$$L_0 = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \tau(r) \frac{\partial}{\partial r} \right) + \frac{\tau(r)}{r^2} L_{\theta\varphi} + V(r) \quad (11.3)$$

eq11.2b

where

$$L_{\theta\varphi} = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2},$$

pr:Lthph1

which is the centrifugal term from equation 11.2. In the next few sections we will study the properties of L_0 .

11.2 Discussion of $L_{\theta\varphi}$

Note that $L_{\theta\varphi}$ is a hermitian operator on the surface of the sphere, as shown by the following argument. In an earlier chapter we derived the Green's Identity

$$\int d^3x S^*(\mathbf{x}) L_0 u(\mathbf{x}) = \int d^3x (u^*(\mathbf{x}) L_0 S(\mathbf{x}))^* \quad (11.4)$$

eq11.3

28 Mar p5

where u and S satisfy RBC. We use this fact to show the hermiticity of $L_{\theta\varphi}$. Consider the functions

$$S(\mathbf{x}) = S(r)S(\theta, \varphi) \quad \text{and} \quad u(\mathbf{x}) = u(r)u(\theta, \varphi)$$

where u and S satisfy RBC. Such functions are a subset of the functions which satisfy equation 11.4. Choose $u(\theta, \varphi)$ and $S(\theta, \varphi)$ to be periodic in the azimuthal angle φ :

$$u(\theta, \varphi) = u(\theta, \varphi + 2\pi), \quad S(\theta, \varphi) = S(\theta, \varphi + 2\pi).$$

Now substitute $d^3x = r^2 dr d\Omega$ and L_0 (as defined in equation 11.3) into equation 11.4. The term

$$-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \tau(r) \frac{\partial}{\partial r} \right) + V(r)$$

in L_0 is hermitian so it cancels out in 11.4. All that is left is

$$\begin{aligned} \int r^2 dr S^*(r) \frac{\tau(r)}{r^2} u(r) \int d\Omega S^*(\theta, \varphi) L_{\theta\varphi} u(\theta, \varphi) = \\ \int r^2 dr S^*(r) \frac{\tau(r)}{r^2} u(r) \int d\Omega (u^*(\theta, \varphi) L_{\theta\varphi} S(\theta, \varphi))^* \end{aligned}$$

This can be rewritten as

28 Mar p6

$$\begin{aligned} \int r^2 dr S^*(r) \frac{\tau(r)}{r^2} u(r) d\Omega \left[\int S^*(\theta, \varphi) L_{\theta\varphi} u(\theta, \varphi) \right. \\ \left. - \int d\Omega (u^*(\theta, \varphi) L_{\theta\varphi} S(\theta, \varphi))^* \right] = 0. \end{aligned}$$

The bracket must then be zero. So

$$\int d\Omega (S^*(\theta, \varphi) L_{\theta\varphi} u(\theta, \varphi)) = \int d\Omega (u^*(\theta, \varphi) L_{\theta\varphi} S(\theta, \varphi))^*. \quad (11.5)$$

This is the same as equation 11.4 with $d^3x \rightarrow d\Omega$ and $L_0 \rightarrow L_{\theta\varphi}$. Thus *eq11.4* $L_{\theta\varphi}$ is hermitian. If the region did not include the whole sphere, we just integrate the region of physical interest and apply the appropriate boundary conditions. Equation 11.5 can also be obtained directly from *28 Mar p7* the form of $L_{\theta\varphi}$ by applying integration by parts on $L_{\theta\varphi}$ twice, but using $L_0 = L_0^*$ is much more elegant.

Note that the operators $\frac{\partial^2}{\partial \varphi^2}$ and $L_{\theta\varphi}$ commute:

$$\left[\frac{\partial^2}{\partial \varphi^2}, L_{\theta\varphi} \right] = 0.$$

Thus we can reduce equation 11.1 to a one dimensional case and expand the Green's function G in terms of a single set of eigenfunctions which are valid for both $-\partial^2/\partial \varphi^2$ and $L_{\theta\varphi}$. We know that L_0 and $L_{\theta\varphi}$ are hermitian operators, and thus the eigenfunctions form a complete set. For this reason this method is valid.

11.3 Spherical Eigenfunctions

28 Mar p8

We want to find a common set of eigenfunctions valid for both $L_{\theta\varphi}$ and $-\partial^2/\partial\varphi^2$. Note that

$$-\frac{\partial^2}{\partial\varphi^2} \frac{e^{im\varphi}}{\sqrt{2\pi}} = m^2 \frac{e^{im\varphi}}{\sqrt{2\pi}} \quad m = 0, \pm 1, \pm 2, \dots$$

pr:sphHarm1 So $(2\pi)^{-1/2}e^{im\varphi}$ are normalized eigen functions of $-\partial^2/\partial\varphi^2$. We define the functions $Y_l^m(\theta, \varphi)$ as the set of solutions to the equation

$$L_{\theta\varphi} Y_l^m(\theta, \varphi) = l(l+1)Y_l^m(\theta, \varphi) \quad (11.6)$$

eq11.5 for eigen values $l(l+1)$ and periodic boundary conditions, and the equation

$$-\frac{\partial^2}{\partial\varphi^2} Y_l^m(\theta, \varphi) = m^2 Y_l^m(\theta, \varphi) \quad m = 0, \pm 1, \pm 2, \dots \quad (11.7)$$

eq11.6 We can immediately write down the orthogonality condition (due to the hermiticity of the operator $L_{\theta\varphi}$):

pr:orthon3
is this true?

$$\int d\Omega Y_l^{m*}(\theta, \varphi) Y_{l'}^{m'}(\theta, \varphi) = \delta_{l,l'} \delta_{m,m'}.$$

(If there are degeneracies, we may use Gram–Schmidt techniques to arrive at this result). We can choose the normalization coefficient to be one. There is also a completeness relation which will be given later.

11.3.1 Reduced Eigenvalue Equation

28 Mar p9

pr:ulm1

We now separate the eigenfunction into the product of a φ -part and a θ -part:

$$Y_l^m(\theta, \varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} u_l^m(\cos\theta) \quad (11.8)$$

eq11.6b (which explicitly solves equation 11.7, the differential equation involving φ) so that we may write equation 11.6 as

$$\left[-\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{m^2}{\sin^2\theta} \right] u_l^m(\cos\theta) = l(l+1)u_l^m(\cos\theta).$$

All we have left to do is solve this eigenvalue equation. The original region was the surface of the sphere because the solid angle represents area on the surface. We make a change of variables:

$$x = \cos \theta.$$

The derivative operator becomes

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \frac{d}{dx}$$

so

$$-\frac{1}{\sin \theta} \frac{d}{d\theta} = \frac{d}{dx}.$$

The eigen value equation for u becomes

$$\left[-\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) + \frac{m^2}{1-x^2} \right] u_l^m(x) = l(l+1)u_l^m(x) \quad (11.9)$$

defined on the interval $-1 < x < 1$. Thus $x = 1$ corresponds to $\theta = 0$, *eq11.7* and $x = -1$ corresponds to $\theta = \pi$. Note that τ , which represents the effective tension, is proportional to $1 - x^2$, so both end points are singular points. On account of this we get both regular and irregular *28 Mar p10* solutions. A solution occurs only if the eigen value l takes on a special value. Requiring regularity at $x = \pm 1$ implies l is an integer. Note also *Verify this* that equation 11.9 represents an infinite number of one dimensional eigenvalue problems (indexed by m), which makes sense because we started with a partial differential equation eigenvalue problem.

The way to solve the equation near a singular point is to look for solutions of the form x^p ·[power series], as in the solutions to Bessel's *change this* equation. *30 Mar p1*

11.3.2 Determination of $u_l^m(x)$

We now determine the function $u_l^m(x)$ which is regular at $x = \pm 1$. Suppose that it is of the form

30 Mar p2

$$u_l^m(x) = (1-x^2)^\beta (\text{power series}). \quad (11.10)$$

We want to determine the power term β . First we compute

eq11.8

$$(1-x^2)\frac{d}{dx}(1-x^2)^\beta = \beta(-2x)(1-x^2)^\beta,$$

and then

$$\begin{aligned} \frac{d}{dx} \left[(1-x^2)\frac{d}{dx}(1-x^2)^\beta \right] &= \beta^2(1-x^2)^{\beta-1}(-2x)^2 \\ &\quad + \beta(-2x)(1-x^2)^\beta + \dots \end{aligned}$$

For the case $x \rightarrow 1$ we can drop all but the leading term:

$$\frac{d}{dx} \left[(1-x^2)\frac{d}{dx}(1-x^2)^\beta \right] \approx 4x^2\beta^2(1-x^2)^{\beta-1} \quad x \rightarrow 1. \quad (11.11)$$

eq11.9

Plugging equations 11.10 and 11.11 into equation 11.9 gives

$$(1-x^2)^{\beta-1}A[-4\beta^2 + m^2] \approx (l+1)lA(1-x^2)^\beta, \quad x \rightarrow 1.$$

where A is the leading constant from the power series. Note however that $(1-x^2)^\beta$ approaches zero faster than $(1-x^2)^{\beta-1}$ as $x \rightarrow 1$. So we get $m^2 = 4\beta^2$, or

$$\beta = \pm \frac{m}{2}.$$

We thus look for a solution of the form

$$u_l^m(x) = (1-x^2)^{m/2}C_m(x), \quad (11.12)$$

eq11.9a

30 Mar p3

where $C_m(x)$ is a power series in x with implicit l dependence. We expect regular and irregular solutions for $C_m(x)$. We plug this equation into equation 11.9 to get an equation for $C_m(x)$. The result is

$$-(1-x^2)C_m'' + 2x(m+1)C_m' - (l-m)(l+m+1)C_m(x) = 0. \quad (11.13)$$

eq11.9b

We still have the boundary condition that $u_l^m(x)$ is finite. In the case that $m = 0$ we have

$$-(1-x^2)C_0'' + 2xC_0' - l(l+1)C_0 = 0. \quad (11.14)$$

eq11.10

pr:LegEq1

This is called Legendre's equation. We want to find the solution of this equation which is regular at $x = 1$. We define $P_l(x)$ to be such

a solution. The irregular solution at $x = 1$, called $Q(x)$, is of interest if the region R in our problem excludes $x = 1$ (which corresponds to $\cos \theta = 1$ or $\theta = 0$). Note that we consider l to be an arbitrary complex number. (From the “general theory”, however, we know that the eigenvalues are real.) By convention we normalize: $P_l(1) = 1$. We know $C_0(x) = P_l(x)$, because we defined $C_0(x)$ to be regular at $x = 1$. But $x = -1$ is also a singular point. We define $\tilde{R}_l(x)$ to be the regular solution and $\tilde{I}_l(x)$ to be the irregular solution at $x = -1$. We can write

$$C_0(x) = P_l(x) = A(l)\tilde{R}_l(x) + B(l)\tilde{I}_l(x). \quad (11.15)$$

But if we further require that P_l must be finite (regular) at $x = -1$, we then have $B(l) = 0$ for $l = 0, 1, 2, \dots$

30 Mar p4

We now take Legendre's equation, 11.14, with $C_0(x)$ replaced by $P_l(x)$ (the regular solution), and differentiate it m times using Leibnitz formula

pr:LeibForm1

$$\frac{d^m}{dx^m} (f(x)g(x)) = \sum_{i=0}^m \binom{m}{i} \frac{d^i f}{dx^i} \frac{d^{m-i} g}{dx^{m-i}}.$$

This yields

$$\begin{aligned} & -(1-x^2) \frac{d^2}{dx^2} \left(\frac{d^m}{dx^m} P_l(x) \right) + 2(m+1)x \frac{d}{dx} \left(\frac{d^m}{dx^m} P_l(x) \right) \\ & - (l-m)(l+m+1) \frac{d^m}{dx^m} P_l(x) = 0. \end{aligned} \quad (11.16)$$

Thus $(d^m/dx^m)P_l(x)$ is also a solution of equation 11.13. We thus see that

eq11.11

$$C_m(x) = \alpha \frac{d^m}{dx^m} P_l(x), \quad (11.17)$$

where α still needs to be determined. Once we find out how to choose l so $P_l(x)$ is 0 at l , $P_{l'}$ for $l' \neq l$ will also be zero. So we see that once we determine constraints on l such that the $P_l(x)$ which solves the $m = 0$ equation is zero at zero at $x = \pm 1$, we can generate a solution for the case $m \neq 0$.

eq11.11b

We now calculate a recurrence relation for $P_l(x)$. Set $x = 1$ in equation 11.16. This gives

pr:recrell

$$2(m+1) \left[\frac{d^{m+1}}{dx^{m+1}} P_l(x) \right]_{x=1} = (l-m)(l+m+1) \left[\frac{d^m}{dx^m} P_l(x) \right]_{x=1}. \quad (11.18)$$

This tells us the $(m + 1)$ th derivative of $P_l(x)$ in terms of the m th derivative. We can differentiate l times if l is an integer. Take l to be an integer. The case $m = l$ yields

$$\left[\frac{d^{l+1}}{dx^{l+1}} P_l(x) \right]_{x=1} = 0. \quad (11.19)$$

ask Baker isn't So all derivatives are zero for $m > l$ at $x = 1$. This means that $P(x)$ this valid only is an l th order polynomial, since all of its Taylor coefficients vanish for at $x = 1$ $m > l$. Since P_l is regular at $x = 1$ and is a polynomial of degree l , 30 Mar p5 it must be regular at $x = -1$ also. If l were not an integer, we would obtain a series which diverges at $x = \pm 1$. Thus we conclude that l must be an integer. Note that even for the solution which is not regular at $x = \pm 1$, for which l is not an integer, equation 11.18 is still valid for calculating the series.

For a general m , we substitute equation 11.17 into equation 11.12 to get

$$u_l^m(x) = \alpha(1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x). \quad (11.20)$$

This equation holds for $m = 0, 1, 2, \dots$. Furthermore this equation solves equation 11.13. Note that because $\frac{d^m}{dx^m} P_l(x)$ is regular at $x = 1$ and $x = -1$, $u_l^m(x)$ is also. 30 Mar p6

We now compute the derivative. Using equation 11.18, for $m = 0$ we get

$$2 \frac{d}{dx} P_l(x)|_{x=1} = l(l+1)P_l(1) = l(l+1).$$

By repeating this process for $m = 1, 2, \dots$ and using induction, we find yet to be verified that the following polynomial satisfies equation 11.18

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \quad (11.21)$$

eq11.13 This is called Rodrigues formula for the Legendre function.
pr:rodform1 We define the associated Legendre polynomial

$$\begin{aligned} P_l^m(x) &= (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) & m \geq 0 \\ &= \frac{(1 - x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l, & m \geq 0. \end{aligned} \quad (11.22)$$

For $m > l$, $P_l^m(x) = 0$. So the allowed range of m is $-l \leq m \leq l$. Thus the value of m affects what the lowest eigenvalue, $l(l+1)$, can be.

11.3.3 Orthogonality and Completeness of $u_l^m(x)$

We want to choose $u_l^m(x)$ to be normalized. We define the normalized eigen functions as the set of eigen functions which satisfies the condition (with $\sigma = 1$)

$$\int_{-1}^1 dx (u_l^m(x))^* u_{l'}^m(x) = 1. \quad (11.23)$$

We need to evaluate $\int_{-1}^1 dx |P_l^m(x)|^2$. Using integration by parts and equation 11.22 we get (a short exercise)

$$\int_{-1}^1 dx |P_l^m(x)|^2 = \int_{-1}^1 dx (P_l^m(x))^* P_l^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}, \quad (11.24)$$

so the normalized eigenfunctions are

$$u_l^m(x) = \sqrt{\frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}} P_l^m(x). \quad (11.25)$$

The condition for orthonormality is

$$\langle u_l^m(x), u_{l'}^m(x) \rangle = \int_{-1}^1 dx (u_l^m(x))^* u_{l'}^m(x) = \delta_{ll'}, \quad (11.26)$$

The corresponding completeness relation is (as usual, with $\sigma = 1$)

$$\sum_{l=m}^{\infty} u_l^m(x) u_l^m(x') = \delta(x - x'). \quad (11.27)$$

The problem we wanted to solve was equation 11.6, so we substitute back in $x = \cos \theta$ into the completeness relation, which gives

$$\begin{aligned} \sum_{l=m}^{\infty} u_l^m(\cos \theta) u_l^m(\cos \theta') &= \delta(\cos \theta - \cos \theta') \\ &= \delta((\theta - \theta')(-\sin \theta)) \\ &= \frac{\delta(\theta - \theta')}{\sin \theta}. \end{aligned}$$

In the second equality we used a Taylor expansion for θ near θ' , which yields

$$\cos \theta - \cos \theta' = -(\theta - \theta') \sin \theta.$$

In the third equality we used the δ -function property $\delta(ax) = |a|^{-1}\delta(x)$. The completeness condition for $u_l^m(\cos \theta)$ is thus

$$\sum_{l=m}^{\infty} u_l^{m*}(\cos \theta) u_l^m(\cos \theta') = \frac{\delta(\theta - \theta')}{\sin \theta}. \quad (11.28)$$

30 Mar p8

Similarly, the orthogonality condition becomes (since $\int_{-1}^1 d(\cos \theta) = \int_0^\pi \sin \theta d\theta$)

$$\int_0^\pi d\theta \sin \theta u_l^m(\cos \theta) u_{l'}^m(\cos \theta) = \delta_{ll'}. \quad (11.29)$$

11.4 Spherical Harmonics

pr:spherH1

1 Apr p1a

We want to determine the properties of the functions Y_l^m , such as completeness and orthogonality, and to determine their explicit form. We postulated that the solution of equations 11.6 and 11.7 has the form (c.f., equation 11.8)

$$Y_l^m(\theta, \varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} u_l^m(\cos \theta)$$

for integer $l = m, m+1, m+2, \dots$ and $m \geq 0$, where we have found (equation 11.25)

$$u_l^m(\cos \theta) = \sqrt{\frac{(l-m)!}{(l+m)!} \frac{2l+1}{2}} (\sin \theta)^m \left(\frac{d}{d \cos \theta} \right)^m P_l(\cos \theta).$$

We define the $Y_l^{-m}(\theta, \varphi)$, for $m > 0$, as

$$\begin{aligned} Y_l^{-m}(\theta, \varphi) &\equiv (-1)^m Y_l^m(\theta, \varphi)^* \\ &= (-1)^m \frac{e^{-im\varphi}}{\sqrt{2\pi}} u_l^m(\cos \theta). \end{aligned}$$

The term $(-1)^m$ is a phase convention and $\frac{e^{-im\varphi}}{\sqrt{2\pi}}$ is an eigen function. This is often called the Condon-Shortley phase convention.

pr:consho1

11.4.1 Othonormality and Completeness of Y_l^m

We saw that the functions u_l^m satisfy the following completeness condition:

$$\sum_{l=m}^{\infty} u_l^m(\cos \theta) u_l^m(\cos \theta') = \frac{\delta(\theta - \theta')}{\sin \theta} \quad \text{for all } m$$

where m is fixed and positive. We know that

$$\sum_{m=-\infty}^{\infty} \frac{e^{im\varphi'}}{\sqrt{2\pi}} \frac{e^{-im\varphi}}{\sqrt{2\pi}} = \delta(\varphi - \varphi'). \quad (11.30)$$

Multiply $\frac{1}{\sin \theta} \delta(\theta - \theta')$ into equation 11.30, so that

eq11.15

1 GApr 1b

$$\begin{aligned} \frac{\delta(\varphi - \varphi') \delta(\theta - \theta')}{\sin \theta} &= \sum_{m=-\infty}^{\infty} \frac{e^{im\varphi}}{\sqrt{2\pi}} \left(\sum_{l \geq |m|}^{\infty} u_l^{|m|}(\cos \theta) u_l^{|m|*}(\cos \theta) \right) \frac{e^{-im\varphi'}}{\sqrt{2\pi}} \\ &= \sum_{m=-\infty}^{\infty} \sum_{l \geq |m|}^{\infty} Y_l^m(\theta, \varphi) Y_l^{m*}(\theta, \varphi'), \end{aligned}$$

since

$$Y_l^m = (-1)^m \frac{e^{im\varphi}}{\sqrt{2\pi}} u_l^{|m|}(\cos \theta) \quad \text{for } m < 0,$$

and

$$Y_l^{m*} = (-1)^m \frac{e^{-im\varphi}}{\sqrt{2\pi}} u_l^{|m|}(\cos \theta) \quad \text{for } m < 0.$$

Thus we have the completeness relation

$$\delta(\Omega - \Omega') = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi'). \quad (11.31)$$

We also note that $L_{\theta\varphi}$ has $(2l+1)$ -fold degenerate eigenvalues $l(l+1)$ in

eq11.16

$$L_{\theta\varphi} Y_l^m(\theta, \varphi) = l(l+1) Y_l^m(\theta, \varphi).$$

Thus m is like a degeneracy index in this equation.

Next we look at the orthogonality of the spherical harmonics. The pr:orthon5

orthogonality relation becomes

$$\begin{aligned} \int d\Omega Y_l^{m'}(\theta, \varphi) Y_l^m(\theta, \varphi) &= \int_0^{2\pi} \frac{d\varphi}{2\pi} \delta_{mm'} \int_{-1}^1 d\cos\theta u_l^m(\cos\theta) u_l^{m'}(\cos\theta) \\ &= \delta_{ll'} \delta_{mm'}, \end{aligned}$$

where $d\Omega = d\varphi d\theta \sin\theta$ on the right hand side, Because the u 's are orthogonal and the $e^{-im\varphi}$'s are orthogonal, the right hand side is zero when $l \neq l'$ or $m \neq m'$.

11.5 GF's for Spherical Symmetry

1 Apr 2a

We now want to solve the Green's function problem for spherical symmetry.

11.5.1 GF Differential Equation

The first step is to convert the differential equation into spherical coordinates. The equation we are considering is

$$[L_0 - \lambda\sigma(\mathbf{x})]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}'). \quad (11.32)$$

eq11.16a

I don't know how to fix this.

By substituting the L_0 for spherically symmetric problems, which we found in equation 11.3, we have

$$\begin{aligned} &\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \tau(r) \frac{d}{dr} \right) + \frac{\tau(r)}{r^2} L_{\theta\varphi} + V(r) - \lambda\sigma \right] G_{lm}(\mathbf{x}, \mathbf{x}'; \lambda') \\ &= \frac{\delta(r - r')}{r^2} \delta(\Omega - \Omega') \\ &= \frac{\delta(r - r')}{r^2} \sum_{l=0}^{\infty} \sum_{m=l}^l Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi'). \end{aligned} \quad (11.33)$$

eq11.17

pr:ExpThm2

where the second equality follows from the completeness relation, equation 11.31. Thus we try the solution form

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_{l=0}^{\infty} \sum_{m=l}^l Y_l^m(\theta, \varphi) G_{lm}(r, r'; \lambda') Y_l^{m*}(\theta', \varphi'). \quad (11.34)$$

eq11.17b Note that the symmetry of θ, ϕ and θ', ϕ' in this solution form means that Green's reciprocity principle is satisfied, as required. Substituting this into equation 11.33 and using equation 11.6 results in $L_{\theta\phi}$ being replaced by the eigenvalue of Y_l^m , which is $l(l+1)$. Superposition says that we can look at just one term in the series. Since the linear operator no longer involves θ, ϕ , we may divide out the Y_l^m 's from both sides to get the following radial equation

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \tau(r) \frac{d}{dr} \right) + \frac{\tau(r)}{r^2} l(l+1) + V(r) \right] G_{lm}(r, r'; \lambda') = \frac{\delta(r - r')}{r^2}. \quad (11.35)$$

The linear operator for this equation has no m dependence. That is, m eq11.18 is just a degeneracy index for the $2l+1$ different solutions of the $L_{\theta\phi}$ equation for fixed l . Thus we can rewrite our radial Green's function G_{lm} as G_l and define the radial operator as

$$L_0^{(l)} = -\frac{d}{dr} \left(r^2 \tau(r) \frac{d}{dr} \right) + r^2 \left[\frac{\tau(r)}{r^2} l(l+1) + V(r) \right]. \quad (11.36)$$

We have reduced the three dimensional Green's function to the standard eq11.18b single dimensional case with effective tension $r^2 \tau(r)$, effective potential energy $V(r)$, and a centripetal kinetic energy term $(\tau(r)/r^2)l(l+1)$.

11.5.2 Boundary Conditions

What about the boundary conditions? If the boundary conditions are pr:bc5
1 Apr 2b not spherically symmetric, we need to take account of the angles, i.e.,

$$[\hat{n} \cdot \nabla + \kappa(S)]G(\mathbf{x} \mathbf{x}', \lambda) = 0,$$

for \mathbf{x} on S and \mathbf{x}' in R .

Consider a spherical region as shown in figure 11.2. For spherically fig11b symmetric boundary conditions, we can set $\kappa_{int}(S) = k_a$ and $\kappa_{ext}(S) = k_b$. Thus

$$\begin{aligned} \left[-\frac{\partial}{\partial r} + k_a \right] G &= 0 \quad \text{for } r = a, \\ \left[\frac{\partial}{\partial r} + k_b \right] G &= 0 \quad \text{for } r = b. \end{aligned}$$

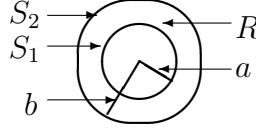


Figure 11.2: The general boundary for spherical symmetry.

If we insert G from equation 11.34 into these conditions, we find the following conditions on how G_l behaves:

$$\left[-\frac{\partial}{\partial r} + k_a \right] G_l = 0 \quad \text{for } r = a,$$

$$\left[\frac{\partial}{\partial r} + k_b \right] G_l = 0 \quad \text{for } r = b,$$

for all l . These equations, together with equation 11.35, uniquely determine G_l .

With these definitions we can examine three interesting cases:

- (1) the internal problem $a \rightarrow \infty$,
- (2) the external problem $b \rightarrow 0$, and
- (3) the all space problem $a \rightarrow 0$ and $b \rightarrow \infty$.

pr:spcProb1

These cases correspond to bound state, scattering, and free space problems respectively.

11.5.3 GF for the Exterior Problem

4 Apr p1

We will now look at how to determine the radial part of the Green's function for the exterior problem. essential idea is that we have taken a single partial differential equation and broken it into several ordinary differential equations. For the spherical exterior problem the region R of interest is the region outside a sphere of radius a , and the boundary S is the surface of the sphere. The physical parameters are all spherically

symmetric: $\tau(r)$, $\sigma(r)$, $V(r)$, and $\kappa(S) = k_a$. Our boundary condition is

$$\left[-\frac{\partial}{\partial r} + k_a \right] G(\mathbf{x}, \mathbf{x}'; \lambda) = 0,$$

where $r = a$ for all θ, φ (that is, $|\mathbf{x}'| > |\mathbf{x}| = a$). This implies

4 Apr p2,3

$$\left[-\frac{\partial}{\partial r} + k_a \right] G_l(r, r'; \lambda) = 0 \quad \text{for } r' > r = a.$$

The other boundary condition is that G_l is bounded as $r \rightarrow \infty$.

We now want to solve $G_l(r, r'; \lambda)$. Recall that we have seen two ways of expressing the Green's function in terms of solutions of the homogeneous equation. One way is to write the Green's functions as a product of the solution satisfying the upper boundary condition and the solution satisfying the lower boundary condition, and then divide by the Wronskian to ensure continuity. Thus we write

$$G_l(r, r'; \lambda) = -\frac{1}{r^2 \tau(r)} \frac{u_1^l(r_<, \lambda) u_2^l(r_>, \lambda)}{W(u_1^l, u_2^l)}, \quad (11.37)$$

where u_1^l and u_2^l satisfy the equations

eq11.18c

$$[L_0^l - \lambda \sigma(r) r^2] u_1^l(r, \lambda) = 0,$$

$$[L_0^l - \lambda \sigma(r) r^2] u_2^l(r, \lambda) = 0,$$

and the boundary conditions

$$\left[-\frac{\partial}{\partial r} + k_a \right] u_1^l = 0 \quad \text{for } r = a,$$

$$u_2^l(r, \lambda) < \infty \quad \text{when } r \rightarrow \infty.$$

The other way of expressing the Green's function is to look at how it behaves near its poles (or branch cut) and consider it as a sum of residues. This analysis was performed in chapter 4 where we obtained the following bilinear sum of eigenfunctions:

4 April p4

$$G_l(r, r', \lambda) = \sum_n \frac{u_n^{(l)}(r) u_n^{(l)}(r')}{\lambda_n^{(l)} - \lambda}. \quad (11.38)$$

Note that what is meant here is really a generic sum which can mean *eq11.19* either a sum or an integral depending on the spectrum of eigenvalues. For the external problem we are considering, the spectrum is a pure continuum and sums over n should be replaced by integrals over λ_n .

The $u_n^{(l)}(r)$ solve the corresponding eigen value problem

$$L_0^{(l)} u_n^{(l)}(r) = \lambda_n^{(l)} r^2 \sigma(r) u_n^{(l)}(r)$$

with the boundary conditions that

$$\left[-\frac{\partial}{\partial r} + k_a \right] u_n^{(l)} = 0 \quad \text{for } r = a,$$

and $u_n^{(l)}$ is finite as $r \rightarrow \infty$. The interior problem, with $u_n^{(l)}$ finite as $r \rightarrow 0$, has a discrete spectrum.

The normalization of the $u_n^{(l)}(r)$ is given by the completeness relation

$$\sum_n u_n^{(l)}(r) u_n^{(l)}(r') = \frac{\delta(r - r')}{r^2 \sigma(r)}.$$

4 Apr p5

We insert equation 11.38 into 11.34 to get

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_n \frac{u_n^{(l,m)}(\mathbf{x}) u_n^{(l,m)}(\mathbf{x}')}{\lambda_n^{(l)} - \lambda} \quad (11.39)$$

eq11.20

where

$$u_n^{(l,m)}(x) = Y_l^m(\theta, \varphi) u_n^{(l)}(r).$$

and $\lambda_n^{(l)}$ is the position of the n th pole of G_l . So the eigenvalues λ_n are the $\lambda_n^{(l)}$ determined from the r -space eigenvalue problem. The corresponding eigenfunctions $u_n^{(l,m)}(\mathbf{x})$ satisfy

$$L_0^{(l)} u_n^{(l,m)}(\mathbf{x}) = \lambda_n^{(l)} r^2 \sigma(r) u_n^{(l,m)}(\mathbf{x}).$$

The completeness relation for $u_n^{(l,m)}(\mathbf{x})$ is found by substituting equation 11.39 into 11.32 and performing the same analysis as in chapter 4. The result is

$$\sum_n u_n^{(l)}(r) u_n^{(l)}(r') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sigma(\mathbf{x})}.$$

11.6 Example: Constant Parameters

We now look at a problem from the homework. We apply the above analysis to the case where $V = 0$ and τ and σ are constant. Our operator for L becomes (c.f., equation 11.36)

$$L_0^{(l)} = \tau \left[-\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + l(l+1) \right].$$

The equation for the Green's function becomes (after dividing by τ):

$$\left[-\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + l(l+1) - k^2 r^2 \right] G_l(r, r'; \lambda) = \frac{1}{\tau} \delta(r - r'),$$

where $k^2 = \lambda\sigma/\tau = \lambda/c^2$.

11.6.1 Exterior Problem

We again have the boundary conditions

$$\left[-\frac{\partial}{\partial r} + k_a \right] G_l(r, r'; \lambda) = 0, \quad (11.40)$$

where $r = a$, $r' > a$, and G_l bounded as $r \rightarrow \infty$. As usual, we assume (eq11.21d the solution form

$$G_l(r, r', \lambda) = -\frac{1}{r^2 \tau(r)} \frac{u_1^{(l)}(r_<, \lambda) u_2^{(l)}(r_>, \lambda)}{W(u_1^{(l)}, u_2^{(l)})}. \quad (11.41)$$

We solve for u_1 and u_2 :

eq11.21a

$$\left[-\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + l(l+1) - k^2 r^2 \right] u_{1,2}^{(l)} = 0,$$

4 Apr p7

where u_1 and u_2 satisfy the boundary conditions as $r = a$ and $r \rightarrow \infty$ respectively and $k \equiv \omega/c = \sqrt{\lambda/c^2}$. This is the spherical Bessel equation from the third assignment of last quarter. We found the solution

$$u(r) = \frac{R(r)}{\sqrt{r}},$$

where $R(r)$ satisfies the regular Bessel equation

$$\left[-\frac{d}{dr} \left(r \frac{d}{dr} \right) + \frac{(l + \frac{1}{2})^2}{r} - k^2 r \right] R(r) = 0.$$

The solutions for this equation are:

$$R \sim J_{l+\frac{1}{2}}, N_{l+\frac{1}{2}}.$$

pr:sphBes

By definition

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x),$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x),$$

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}(x),$$

where $j_l(kr)$ is a spherical Bessel function, $n_l(kr)$ is a spherical Neumann function, and $h_l^{(1)}(kr)$ is a spherical Hankel function. So we can write

$$u_1 = A j_l(kr_{<}) + B n_l(kr_{<}),$$

$$u_2 = h_l^{(1)}(kr_{>}),$$

Is this correct? Note that the u_2 solution is valid because it is bounded for large r :

$$\lim_{x \rightarrow \infty} h_l^{(1)}(x) = -\frac{i}{x} e^{ix} (-i)^l.$$

11.6.2 Free Space Problem

4 Apr p8

We now take the special case where $a = 0$. The boundary condition becomes the regularity condition at $r = 0$, which kills the $n_l(kr_{<})$ solution.

The solutions $u_l^{(1)}$ and $u_l^{(2)}$ are then

$$u_l^{(1)} = j_l(kr),$$

$$u_l^{(2)} = h_l^{(1)}(kr).$$

4 Apr p9

Let λ be an arbitrary complex number. Note that since $h_l^{(1)}(x) =$

$j_l(x) + in_l(x)$, we have

$$W(j_l(x), h_l^{(1)}(x)) = iW(j_l(x), n_l(x)) = \frac{i}{x^2}.$$

The last equality follows immediately if we evaluate the Wronskian for large r using

$$j_l(x) \approx \cos(x - (l+1)\pi/2) \quad x \rightarrow \infty,$$

$$n_l(x) \approx \sin(x - (l+1)\pi/2) \quad x \rightarrow \infty,$$

and recall that τW is a constant (for the general theory, c.f., problem 1, set 3) where “ τ ” is $r^2\tau$ for this problem. In particular we have

$$W(j_l, h_l^{(1)}) = \frac{i}{(kr)^2}.$$

We then get from equation 11.41

$$\begin{aligned} G_l &= \frac{1}{r^2\tau} \frac{j_l(kr_<)h_l^{(1)}(kr_>)}{k \frac{i}{(kr)^2}} \\ &= \frac{ik}{\tau} j_l(kr_<)h_l^{(1)}(kr_>). \end{aligned} \quad (11.42)$$

(eq11.21c

We have thus found the solution of

$$\left[-\nabla^2 - \frac{\lambda}{c^2} \right] G = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\tau}, \quad (11.43)$$

which is the fundamental three-space Green’s function. We found (eq11.22

4 Apr p10

$$G(\mathbf{x}, \mathbf{x}', \lambda) = \frac{ik}{\tau} \sum_{lm} Y_l^m(\theta, \varphi) j_l(kr_<) h_l(kr_>) Y_l^{m*}, \quad (11.44)$$

which is a simple combination of equation 11.34 and 11.42. In the first homework assignment we solve this by a different method to find an explicit form for equation 11.43. Here we show something related. In equation 11.43 we have $G = G(|\mathbf{x} - \mathbf{x}'|)$ since $V = 0$ and σ and τ

eq11.22a

constant. This gives translational and rotational invariance, which corresponds to isotropy and homogeneity of space. We solve by choosing $\mathbf{x} = 0$ so that

$$j_l(kr_<) = j_0(kr) + 0's$$

as $r' \rightarrow 0$. Only the $l = 0$ term survives, since as $r' \rightarrow 0$ we have

4 Apr p11

$$l = 0 \quad \longrightarrow \quad j_l(0) = 1,$$

$$l \neq 0 \quad \longrightarrow \quad j_l(0) = 0.$$

Thus we have

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{ik}{\tau} |Y_0^0|^2 h_0(kr).$$

Since $l = 0$ implies $m = 0$, we get $Y_0^0 = \text{const.} = 1/4\pi$ since it satisfies the normalization $\int d\Omega |Y|^2 = 1$. We also know that

$$h_0^{(1)}(x) = -\frac{i}{x} e^{ix}.$$

This gives us

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{1}{\tau 4\pi r} e^{ikr}.$$

We may thus conclude that

$$G(|\mathbf{x} - \mathbf{x}'|) = \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{4\pi |\mathbf{x} - \mathbf{x}'| \tau}. \quad (11.45)$$

eq11.25

stuff omitted

11.7 Summary

1. The form of the linear operator in spherical coordinates is

$$L_0 = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \tau(r) \frac{\partial}{\partial r} \right) + \frac{\tau(r)}{r^2} L_{\theta\varphi} + V(r),$$

where

$$L_{\theta\varphi} = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial^2 \varphi}.$$

2. $L_{\theta\phi}$ is hermitian.
3. The eigenvalue equations for Y_l^m are

$$L_{\theta\phi} Y_l^m(\theta, \varphi) = l(l+1) Y_l^m(\theta, \varphi),$$

$$-\frac{\partial^2}{\partial \varphi^2} Y_l^m(\theta, \varphi) = m^2 Y_l^m(\theta, \varphi) \quad m = 0, \pm 1, \pm 2, \dots$$

4. The partial wave expansion for the Green's function is

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_{l=0}^{\infty} \sum_{m=l}^l Y_l^m(\theta, \varphi) G_{lm}(r, r'; \lambda') Y_l^{m*}(\theta, \varphi).$$

5. The Green's function for the free spce problem is

$$G(|\mathbf{x} - \mathbf{x}'|) = \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|\tau}.$$

11.8 References

The preferred special functions reference for physicists seems to be [Jackson75]. Another good source is [Arfken85].

This material is developed by example in [Fetter81].

Chapter 12

Steady State Scattering

Chapter Goals:

- Find the free space Green's function outside a circle of radius a due to point source.
- Find the free space Green's function in one, two, and three dimensions.
- Describe scattering from a cylinder.

12.1 Spherical Waves

We now look at the important problem of steady state scattering. Consider a point source at \mathbf{x}' with sinusoidal time dependence

6 Apr p1

pr:sssc1

$$f(\mathbf{x}', t) = \delta(\mathbf{x} - \mathbf{x}')e^{-i\omega t},$$

whose radiated wave encounters an obstacle, as shown in figure 12.1 *fig11.3*

We saw in chapter 1 that the steady state response for free space with a point source at \mathbf{x}' satisfies

$$\left[L_0 - \sigma \frac{\partial^2}{\partial t^2} \right] u_0(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}')e^{-i\omega t},$$

and was solved in terms of the Green's function,

$$u_0(\mathbf{x}, \omega) = G_0(\mathbf{x}, \mathbf{x}', \lambda = \omega^2 + i\epsilon)e^{-i\omega t} \quad (12.1)$$

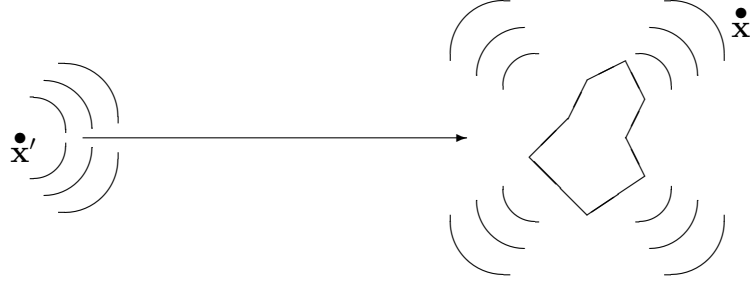


Figure 12.1: Waves scattering from an obstacle.

$$= \frac{e^{ik|\mathbf{x}-\mathbf{x}'|-i\omega t}}{4\pi\tau|\mathbf{x}-\mathbf{x}'|} \quad (12.2)$$

eq11.25b
pr:uoxo1

where the Green's function G_0 satisfies the equation 11.43 and the second equality follows from 11.45. The equation for G_0 can be written

$$[-\nabla^2 - k^2]G_0(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{1}{\tau}\delta(\mathbf{x} - \mathbf{x}') \quad (12.3)$$

eq11.25c
6 Apr p3

with the definition $k = \sqrt{\lambda/c^2} = \omega/c$ (Remember that $\lambda = \omega^2 + i\varepsilon$). We combine these observations to get

$$u_0(\mathbf{x}, \mathbf{x}'; \omega) = \frac{1}{4\pi\tau|\mathbf{x} - \mathbf{x}'|} e^{-i\omega(t-(\mathbf{x}-\mathbf{x}')/c)}.$$

If there is an obstacle (i.e., interaction), then we have a new steady state response

$$u(\mathbf{x}, \mathbf{x}'; \omega) = G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2 + i\varepsilon)e^{-i\omega t},$$

where

$$[L_0 - \lambda\sigma]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') \quad \text{RBC.} \quad (12.4)$$

This is the steady state solution for all time. We used u_0 and G_0 for the free space problem, and u and G for the case with a boundary. Note that equation 12.4 reduces to 12.2 if there is no interaction. The scattered part of the wave is

$$G_S e^{-i\omega t} = (G - G_0)e^{-i\omega t}.$$

pr:GS1
6 Apr p4

12.2 Plane Waves

We now look at the special case of an incident plane wave instead of an incident spherical wave. This case is more common. Note that once we solve the point source problem, we can also solve the plane wave problem, since plane waves may be decomposed into spherical waves. An incident plane wave has the form

$$\Phi_0 = e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$$

where $\mathbf{k} = (\omega/c)\hat{n}$. This is a solution of the homogeneous wave equation pr:Phi0

$$\left[-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Phi_0 = 0.$$

So Φ_0 is the solution to the equation without scattering.

Let Φ be the wave when an obstacle is present,

$$\left[-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Phi = 0$$

which solves the homogeneous wave equation and the regular boundary condition at the surface of the obstacle.

To obtain this plane wave problem, we let \mathbf{x}' go to $-\infty$. We now describe this process. To obtain the situation of a plane wave approaching the origin from $-\infty\hat{z}$, we let $\mathbf{x}' = -r'\hat{z}$, as $r' \rightarrow \infty$. We want to find out what effect this limit has on the plane wave solution we obtained in equation 11.45,

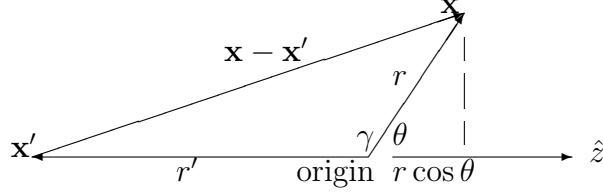
$$G_0 = \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{4\pi\tau|\mathbf{x} - \mathbf{x}'|}.$$

We define the angles γ and θ as shown in figure 12.2. From the figure pr:gamma1
fig12a we see that

$$|\mathbf{x} - \mathbf{x}'| = r' + r \cos \theta = r' - r \cos \gamma, \quad |\mathbf{x}'| \rightarrow \infty.$$

We further recall that the dot product of unit vectors is equal to the cosine of the separation angle,

$$\cos \gamma = \frac{\mathbf{x} \cdot \mathbf{x}'}{rr'} = -\cos \theta.$$

Figure 12.2: Definition of γ and θ ..

So the solution is

$$\begin{aligned}
 G_0(\mathbf{x}, \mathbf{x}'; \lambda) &= \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi\tau|\mathbf{x}-\mathbf{x}'|} \\
 &= \frac{1}{4\pi\tau r'} e^{ik(r-r\cos\gamma)} \\
 &= \frac{1}{4\pi\tau r'} e^{ikr'} e^{ik\mathbf{x}\cdot(-\hat{x}')} \\
 &= \frac{e^{ikr'}}{4\pi\tau r'} e^{i\mathbf{k}\cdot\mathbf{x}}
 \end{aligned}$$

where $\mathbf{k} = \frac{w}{c}(-\hat{x}')$. Note that we have used the first two terms of the approximation in the exponent, but only the first term in the denominator. Thus we see that in this limit the spherical wave u_0 in free space due to a point source is

$$\begin{aligned}
 \lim_{|\mathbf{x}'| \rightarrow \infty} u_0 &= G_0 e^{-i\omega t} \\
 &= \frac{e^{ikr'}}{4\pi r' \tau} \Phi_0
 \end{aligned}$$

where

$$\Phi_0 = e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}.$$

12.3 Relation to Potential Theory

Consider the problem of finding the steady state response due to a point source with frequency ω located at \mathbf{x}' outside a circular region of radius a . The steady state response must satisfy the regular boundary condition

$$\left[-\frac{\partial}{\partial r} G + \kappa_a G \right] = 0 \quad \text{for } r = a. \quad (12.5)$$

In particular we want to find this free space Green's function outside a circle of radius a , where $V = 0$ and σ and τ are constant. eq11A1

The Green's function G satisfies the inhomogeneous wave equation

$$[-\nabla^2 - k^2]G(\mathbf{x}, \mathbf{x}') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\tau}$$

where $k^2 = \lambda\sigma/\tau = \lambda/c^2$. The solution was found to be (from problem 1 of the final exam of last quarter)

$$G = \frac{1}{4\pi\tau} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} [J_m(kr_{<}) + X_m H_m^{(1)}(kr_{<})] H_m^{(1)}(kr_{>}). \quad (12.6)$$

with

$$X_m = -\frac{[kJ'_m(ka) - K_a J_m(ka)]}{[kH_m^{(1)}(ka) - K_a H_m^{(1)}(ka)]}. \quad (12.7)$$

Note that $H_m^{(1)}(kr_{>})$ comes from taking $\text{Im}\sqrt{\lambda} > 0$. If we consider $\text{Im}\sqrt{\lambda} < 0$, we would have $H_m^{(2)}$ instead. All the physics is in the functions X_m . Note that from this solution we can obtain the solution to the free space problem (having no boundary circle). Our boundary condition is then G must be regular at $|\mathbf{x}| = 0$: (Eq.BE) 8 Apr p2

$$G \text{ regular, } |\mathbf{x}| = 0. \quad (12.8)$$

(That is, equation 12.5 becomes 12.8.) How do we get this full space solution? From our solution, let a go to zero. So X_m in equation 12.6 goes to zero as a goes to zero, and by definition $G \rightarrow G_0$. The free space Green's function is then eq11A2

$$G_0 = \frac{i}{4\pi\tau} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} J_m(kr_{<}) H_m^{(1)}(kr_{>}). \quad (12.9)$$

This is the 2-dimensional analog of what we did in three dimensions. We use this to derive the plane wave expression in 3-dimensions. eq11A4

We may now obtain an alternative expression for the free space Green's function in two dimensions by shifting the origin. In particular, we place the origin at \mathbf{x}' . This gives us $r' = 0$, for which

$$J_m(kr') H_m^{(1)}(kr)|_{r'=0} = \begin{cases} H_0^{(1)}(kr) & m = 0 \\ 0 & \text{else.} \end{cases}$$

Thus equation 12.9 reduces to

$$G_0(r) = \frac{i}{4\pi\tau} H_0^{(1)}(kr),$$

which can also be written as

$$G_0(|\mathbf{x} - \mathbf{x}'|) = \frac{i}{4\pi\tau} H_0^{(1)}(k|\mathbf{x}' - \mathbf{x}|). \quad (12.10)$$

eq11A5

This is the expression for the two dimensional free space Green's function. In the process of obtaining it, we have proven the Hankel function addition formula

$$H_0^{(1)}(k|\mathbf{x}' - \mathbf{x}|) = \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} J_m(kr_{<}) H_m^{(1)}(kr_{>}).$$

We now have the free space Green's functions for one, two, and three dimensions:

$$\begin{aligned} G_0^{1D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{i}{2k\tau} e^{ik|x-x'|}, \\ G_0^{2D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{i}{4\pi\tau} H_0^{(1)}(k|\mathbf{x}' - \mathbf{x}|), \\ G_0^{3D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi\tau|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (12.11)$$

eq11A6

pr:fsp2

8 Apr 5

We can interpret these free space Green's functions physically as follows. The one dimensional Green's function is the response due to a plane source, for which waves go off in both directions. The two dimensional Green's function is the cylindrical wave from a line source. The three dimensional Green's functions is the spherical wave from a point source. Note that if we let $k \rightarrow 0$ in each case, we have

$$e^{ik|\mathbf{x}-\mathbf{x}'|} = 1 + ik|\mathbf{x} - \mathbf{x}'|,$$

and thus we recover the correct potential respectively for a sheet of charge, a line charge, and a point charge.

12.4 Scattering from a Cylinder

We consider again the Green's function for scattering from a cylinder, equation 12.6

$$G = \frac{1}{4\pi\tau} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} [J_m(kr_{<}) + X_m H_m^{(1)}(kr_{<})] H_m^{(1)}(kr_{>}). \quad (12.12)$$

What is the physical meaning of $[J_m(kr_{<}) + X_m H_m^{(1)}(kr_{<})]$ in this equation? This gives the field due to a point source exterior to the cylinder:

$$\begin{aligned} u &= G(\mathbf{x}, \mathbf{x}', \lambda = \omega^2 + i\varepsilon) e^{-i\omega t} \\ &= \underbrace{G_0 e^{-i\omega t}}_{u_0} + \underbrace{(G - G_0) e^{-i\omega t}}_{u_s} \end{aligned} \quad (12.13)$$

Note that X_m contains the physics of the boundary condition, eq11A6c

$$\left[-\frac{\partial G}{\partial r} + \kappa_a G \right] = 0 \text{ for } r = a.$$

From equations 12.12 and 12.13 we identify the scattered part of the solution, u_s , as

$$u_s = \frac{e^{-i\omega t}}{4\pi\tau} \sum_{m=0}^{\infty} e^{im(\phi-\phi')} X_m H_m^{(1)}(kr) H_m^{(1)}(kr'). \quad (12.14)$$

So we have expanded the total scattered wave in terms of $H_m^{(1)}$, where X_m gives the m th amplitude. Why are the $r_{>}$ and $r_{<}$ in equation 12.11 but not in equation 12.14? Because there is a singular point at $\mathbf{x} = \mathbf{x}'$ in equation 12.11, but u_s will never have a singularity at $\mathbf{x} = \mathbf{x}'$. (Eq.j)

Now consider the more general case of spherical symmetry: $V(r)$, $\tau(r)$, $\sigma(r)$. If these parameters are constant at large distances,

$$\begin{aligned} V(r) &= 0, \\ \tau(r) &= \tau = \text{constant}, \\ \sigma(r) &= \sigma = \text{constant}, \end{aligned}$$

then at large distances G must have the form of equation 12.11,

$$G_0(|\mathbf{x} - \mathbf{x}'|) = \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{4\pi\tau|\mathbf{x} - \mathbf{x}'|}.$$

We shall see that this formula is basic solution form of quantum mechanical scattering.

12.5 Summary

1. The free space Green's function outside a circle of radius a due to point source is

$$G = \frac{1}{4\pi\tau} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} [J_m(kr_{<}) + X_m H_m^{(1)}(kr_{<})] H_m^{(1)}(kr_{>}).$$

with

$$X_m = -\frac{[kJ'_m(ka) - K_a J_m(ka)]}{[kH_m^{(1)}(ka) - K_a H_m^{(1)}(ka)]}.$$

2. The free space Green's function in one, two, and three dimensions is

$$\begin{aligned} G_0^{1D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{i}{2k\tau} e^{ik|x-x'|}, \\ G_0^{2D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{i}{4\pi\tau} H_0^{(1)}(k|\mathbf{x}' - \mathbf{x}|), \\ G_0^{3D}(|\mathbf{x} - \mathbf{x}'|) &= \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{4\pi\tau|\mathbf{x} - \mathbf{x}'|}. \end{aligned}$$

3. For the problem of scattering from a cylinder, the total response u is easily decomposed into an incident part a scattered part u_s , where u_s contains the coefficient X_l .

12.6 References

See any *old* nuclear or high energy physics text, such as [Perkins87].

Chapter 13

Kirchhoff's Formula

As a further application of Green's functions to steady state problems, let us derive Kirchhoff's formula for diffraction through an aperture. pr:Kirch1 Suppose we have a point source of sound waves of frequency ω at some point \mathbf{x}_0 in the left half plane and at $x = 0$ we have a plane with a hole. We want to find the diffracted wave at a point \mathbf{x} in the right-half plane. fig19a If we look at the screen directly, we see the aperture is the yz -plane at $x = 0$ with a hole of shape σ' as shown in the figure. The solution $G(\mathbf{x}, \mathbf{x}_0; \omega)$ satisfies the equations

$$-[\nabla^2 + k^2]G(\mathbf{x}, \mathbf{x}_0; \omega) = \delta(\mathbf{x} - \mathbf{x}_0),$$

$$\left. \frac{\partial G}{\partial x} \right|_{\substack{x=0 \\ \mathbf{x} \notin \sigma'}} = 0.$$

We will reformulate this problem as an integral equation. This integral equation will have as a kernel the solution in the absence of the hole due to a point source at \mathbf{x}' in the R.H.P. This kernel is the free space Green's function, $G_0(\mathbf{x}, \mathbf{x}'; \omega)$, which satisfies the equations

$$-[\nabla^2 + k^2]G_0(\mathbf{x}, \mathbf{x}'; \omega) = \delta(\mathbf{x} - \mathbf{x}'),$$

$$\left. \frac{\partial G_0}{\partial x} \right|_{\substack{x=0 \\ \text{all } y, z}} = 0.$$

This we solve by the method of images. The boundary condition is pr:MethIm2

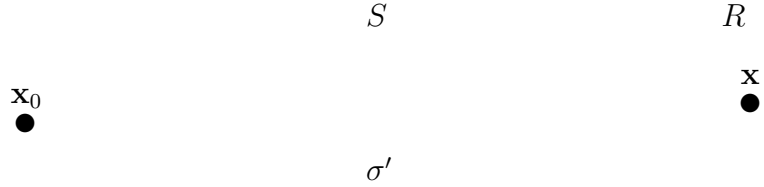


Figure 13.1: A screen with a hole in it.

satisfied by adding an image source at \mathbf{x}'^* , as shown in figure 13.2. Thus the Green's function for this boundary value problem is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \left[\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \right]$$

Now by taking $L_0 = -(\nabla^2 + k^2)$ we may apply Green's second identity

$$\int_{\mathbf{x} \in R} (S^* L_0 u - u L_0 S^*) = \int_{\mathbf{x} \in S} dS \hat{n} \cdot [u \nabla S^* - S^* \nabla u]$$

with $u = G(\mathbf{x}, \mathbf{x}_0)$ and $S^* = G_0(\mathbf{x}, \mathbf{x}')$ where R is the region $x > 0$ and S is the yz -plane. This gives us

$$L_0 u = 0 \quad \text{for } x > 0,$$

$$\left. \frac{\partial u}{\partial x} \right|_{\substack{x=0 \\ x \notin \sigma}} = 0,$$

and

$$L_0 S^* = \delta(\mathbf{x} - \mathbf{x}'),$$

$$\left. \frac{\partial S^*}{\partial x} \right|_{x=0} = 0.$$

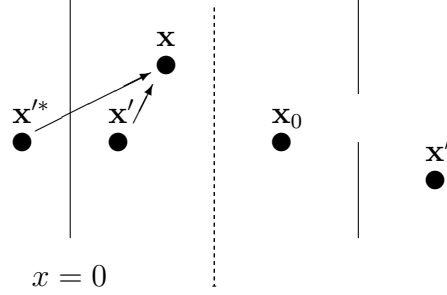


Figure 13.2: The source and image source.

These identities allow us to rewrite Green's second identity as

$$-\int d\mathbf{x} G(\mathbf{x}, \mathbf{x}_0) \delta(\mathbf{x} - \mathbf{x}') = \int dy dz G_0(\mathbf{x}, \mathbf{x}') \left(-\frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0) \right)_{x=0}$$

and therefore

$$G(\mathbf{x}', \mathbf{x}_0) = - \int dy dz G_0(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0) \Big|_{x=0}. \quad (13.1)$$

Thus the knowledge of the disturbance, i.e., the normal component of the velocity at the aperture, determines the disturbance at an arbitrary point \mathbf{x} in the right half plane. We have then only to know $\frac{\partial}{\partial x} G$ at the aperture to know G everywhere.

Furthermore, if \mathbf{x}_0 approaches the aperture, equation 13.1 becomes an integral equation for G for which we can develop approximation methods.

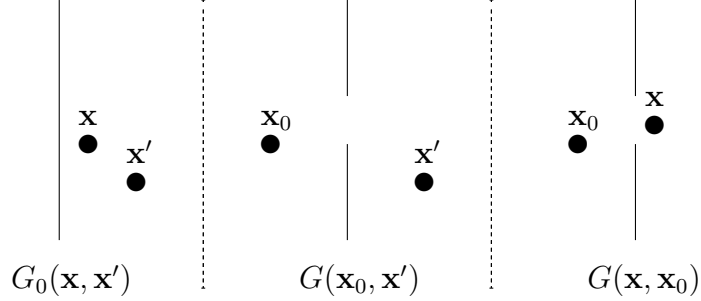
$$G(\mathbf{x}', \mathbf{x}_0) = - \int_{\mathbf{x} \in \sigma'} dy dz G_0(\mathbf{x}, \mathbf{x}') \frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0). \quad (13.2)$$

The configurations for the different G 's are shown in figure 13.3. Note

$$G_0(\mathbf{x}, \mathbf{x}')|_{\mathbf{x}'=0} = \frac{1}{2\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \quad \text{eq19b fig19b}$$

and equation 13.2 becomes

$$G(\mathbf{x}', \mathbf{x}_0) = - \frac{1}{2\pi} \int_{\mathbf{x} \in \sigma'} dy dz \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0) \Big|_{\mathbf{x}=0}.$$

Figure 13.3: Configurations for the G 's.

Now suppose that the size a of the aperture is much larger than the wavelength $\lambda = 2\pi/k$ of the disturbance which determines the distance scale. In this case we expect that the wave in the aperture does not differ much from the undisturbed wave except for within a few wavelengths near the aperture. Thus for $ka \ll 1$ we can write

$$G(\mathbf{x}, \mathbf{x}_0) \approx -\frac{1}{2\pi} \int_{\mathbf{x} \in \sigma'} dy dz \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \left. \frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0) \right|_{\mathbf{x}=0} \left. \frac{\partial}{\partial x} \left| \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \right| \right|_{\mathbf{x}=0},$$

where we have used the substitution

$$\left. \frac{\partial}{\partial x} G(\mathbf{x}, \mathbf{x}_0) \right|_{\mathbf{x}=0} = \left. \frac{\partial}{\partial x} \left| \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \right| \right|_{\mathbf{x}=0}.$$

This equation then gives us an explicit expression for $G(\mathbf{x}', \mathbf{x}_0)$ in terms of propagation from the source at \mathbf{x} to the field point \mathbf{x}' of the velocity disturbance at \mathbf{x} of the velocity distribution $\left. \frac{\partial}{\partial x} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \right|_{\mathbf{x}=0}$ produced by free propagation to \mathbf{x} from the point \mathbf{x}_0 of the disturbance. This yields Huygen's principle and other results of physical optics (Babinet's principle, etc.).

fig19c
pr:Huyg1

13.1 References

See [Fetter80, pp327–332] for a discussion of these results.

Chapter 14

Quantum Mechanics

Chapter Goals:

- State the Green's function equation for the inhomogeneous Schrödinger equation.
- State the Green's function for a bound-state spectra in terms of eigen wave functions.
- State the correspondence between classical wave theory and quantum particle theory.

8 Apr p8
pr:QM1

The Schrödinger equation is

$$\left[H - i\hbar \frac{\partial}{\partial t} \right] \psi(\mathbf{x}, t) = 0$$

where the Hamiltonian H is given by

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}).$$

This is identical to our original equation, with the substitutions $\tau = \hbar^2/2m$, $L_0 = H$, and in the steady state case $\lambda\sigma = E$. For the free space problem, we require that the wave function ψ be a regular function. The expression $|\psi(\mathbf{x}, t)|^2$ is the probability given by the probability amplitude $\psi(\mathbf{x}, t)$. For the time dependent Schrödinger equation, we have the same form as the heat equation, with $\rho c_p \rightarrow i\hbar$. In making the

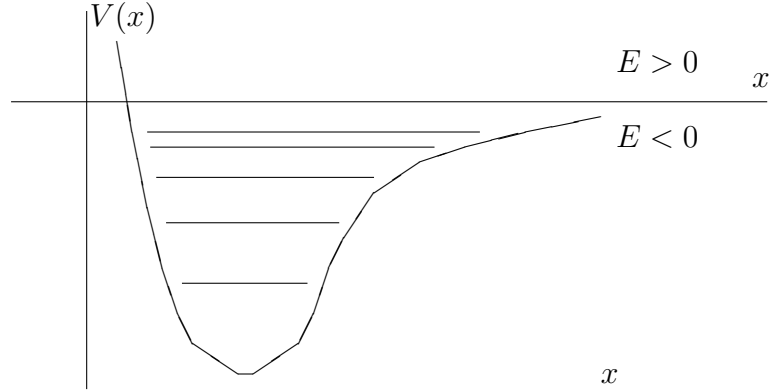


Figure 14.1: An attractive potential.

transition from classical mechanics to quantum mechanics, we use $H = p^2/2m + V(\mathbf{x})$ with the substitution $\mathbf{p} \rightarrow (\hbar/i)\nabla$; this correspondence for momentum means that the better we know the position, and thus the more sharply the wave function falls off, the worse we know the subsequent position. This is the essence of the uncertainty principle.

pr:sss3

We now look at the steady state form. Steady state solutions will be of the form

$$\psi(\mathbf{x}, t) = e^{-i\omega t} \psi_\omega(\mathbf{x}),$$

where $\psi_\omega(\mathbf{x})$ satisfies the equation

$$[H - \hbar\omega] \psi_\omega(\mathbf{x}) = 0.$$

pr:enLev1
8 Apr p9

The allowed energy levels for $\hbar\omega$ are the eigen values E of H :

$$H\psi = E\psi. \quad (14.1)$$

eq13.1

fig13.1

So the allowed frequencies are $\omega = E/\hbar$, for energy eigenvalues E . The energy spectrum can be either discrete or continuous. Consider the potential shown in figure 14.1. For $E = E_n < V(0)$, the energy levels are discrete and there are a finite number of such levels; for $E > V(0)$, the energy spectrum is continuous: any energy above $V(0)$ is allowed. A plot of the complex energy plane for this potential is shown in figure 14.2. The important features are that the discrete energies appear as poles on the negative real axis, and the continuous energies appear as a

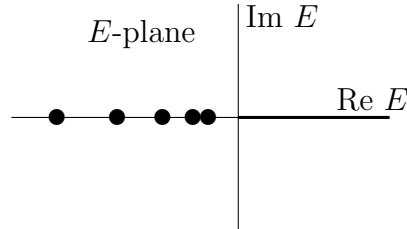


Figure 14.2: The complex energy plane.

branch cut on the positive real axis. Note that where as in this problem there are a finite number of discrete levels, for the coulomb potential there are instead an infinite number of discrete levels. If, on the other hand, we had a repulsive potential, then there would be no discrete spectrum.

The Green's function solves the Schrödinger equation with an inhomogeneous δ -function term:

8 Apr p10

$$(H - E)G(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}') \quad (14.2)$$

where E is a complex variable. The boundary condition of the Green's function for the free space problem is that it be a regular solution. Unlike previously considered problems, in the quantum mechanical problems, the effect of a boundary, (e.g., the surface of a hard sphere) is enforced by an appropriate choice of the potential (e.g., $V = \infty$ for $r > a$). Once we have obtained the Green's function, we can look at its energy spectrum to obtain the ψ 's and E_n 's, using the formula obtained in chapter 4:

eq13.2

$$G(\mathbf{x}, \mathbf{x}; E) = \sum_n \frac{\psi_n(\mathbf{x})\psi_n(\mathbf{x}')}{E_n - E}.$$

This formula relates the solution of equation 14.1 to the solution of equation 14.2.

fig13.2

11 Apr p1

14.1 Quantum Mechanical Scattering

11 Apr p2

11 Apr p3

We now look at the continuum case. This corresponds to the problem of scattering. We use the Green's function to solve the problem of

pr:QMS1

quantum mechanical scattering. In the process of doing this, we will see that the quantum mechanics is mathematically equivalent to the classical mechanics of waves. Both situations involve scattering. The solution u for the classical wave problem is interpreted as a velocity potential, whereas the solution ψ for the quantum mechanical problem is interpreted as the probability amplitude. For the classical wave problem, u^2 is interpreted as intensity, while for quantum mechanics, $|\psi|^2$ is interpreted as probability density. Thus, although the mathematics for these problems is similar, the general difference is in the interpretation.

The case of quantum mechanical scattering is similar to classical scattering, so we consider classical scattering first. We use the Green's function in classical wave theory,

$$[L_0 - \lambda\sigma]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}'),$$

with $\lambda = \omega^2 + i\varepsilon$ for causality, to obtain the steady state response due to a point source,

$$u = e^{-i\omega t}G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2 + i\varepsilon).$$

This steady state solution solves the time dependent classical wave equation,

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2}\right]u(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}'). \quad (14.3)$$

eq13.3

We may decompose the solution u into two parts,

$$u = u_0 + u_{\text{scat}},$$

where

$$u_0 = e^{-i\omega t}G_0 = \frac{e^{-i(\omega t - kR)}}{4\pi\tau R}$$

where $R = |\mathbf{x} - \mathbf{x}'|$, and

$$u_{\text{scat}} = e^{-i\omega t}[G - G_0].$$

pr:uscat1

Note that u_0 is the steady state solution for a point source at $\mathbf{x} = \mathbf{x}'$, solution for a point source at \mathbf{x} , G_0 is the solution for the free case, and u_s is the solution for outgoing scattered waves. Note also that the outgoing scattered waves have no singularity at $\mathbf{x} = \mathbf{x}'$.

14.2 Plane Wave Approximation

If one solves the problem of the scattering of the spherical wave from a point source, that is, for the Green's function problem, then we also have the solution for scattering from a plane wave, merely by letting $|\mathbf{x}'| \rightarrow \infty$. We now define $\Phi(\mathbf{x}, t)$ as the solution of equation 14.3 for the special case in which $\mathbf{x}' \rightarrow -\infty \hat{z}$,

pr:PIWv1
11 Apr p4

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] \Phi(\mathbf{x}, t) = 0. \quad (14.4)$$

For the steady state solution

eq13.4
11 Apr p5

$$\Phi(\mathbf{x}, t) = e^{-i\omega t} \Phi(\mathbf{x}, \omega),$$

we get the equation

$$[L_0 - \sigma \omega^2] \Phi(\mathbf{x}, \omega) = 0.$$

How is this equation solved for positive ω ? There isn't a unique solution for this, just like there wasn't a unique solution for the Green's function. We have already found a solution to this equation by considering the Green's function with the source point going to infinity in the above method, but it does not satisfy the boundary condition appropriate for scattering. 11 Apr p6

In order to determine the unique scattering solution, we must introduce a new boundary condition appropriate for scattering. This is necessary because although our previous equation

$$[L_0 - \sigma \lambda_n] u_n = 0 \quad (14.5)$$

with RBC gave the eigenfunctions, it does not give unique physical solutions for the case of scattering. eq13abc

To find the u_n 's in equation 14.5 we would extract them from the Green's function using, for the discrete case,

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_n \frac{u_n u_n^*}{\lambda_n - \lambda},$$

or for the continuum case,

$$\begin{aligned} u_n &= \frac{1}{2\pi i} [G(\mathbf{x}, \mathbf{x}'; \lambda = \lambda_n + i\varepsilon) - G(\mathbf{x}, \mathbf{x}'; \lambda = \lambda_n - i\varepsilon)] \\ &= \frac{1}{\pi} \text{Im} [G(\mathbf{x}, \mathbf{x}'; \lambda = \lambda_n + i\varepsilon)]. \end{aligned}$$

But these u_n 's are not solutions corresponding to the scattering boundary condition, since they contain both incoming and outgoing waves.

14.3 Quantum Mechanics

11 Apr p7

We now apply what we have said for particles to the case of quantum mechanics. In this case the steady state solutions are of the form

$$\psi = e^{-i(E/\hbar)t} \psi_0.$$

We want the total wave to be a superposition of an incident plane wave and a scattered wave.

$$\psi = e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + \psi_s$$

11 Apr p9

where $\omega = E/\hbar$. Note that $e^{i\mathbf{k}\cdot\mathbf{x}}$ corresponds to an incident plane wave, and ψ_s corresponds to outgoing waves. To get this form, we use the Green's function for the free space problem (no potential)

$$G_0 = \frac{e^{ikR}}{4\pi R\tau} = \frac{m}{2\pi\hbar^2} \frac{e^{ikR}}{R},$$

where

$$k^2 = \frac{\lambda\sigma}{\tau} = \frac{E}{\hbar^2/2m} = \frac{2mE}{\hbar^2} = \frac{p^2}{\hbar^2}.$$

Take the limit $|\mathbf{x}'| \rightarrow \infty$, the free space Green's function becomes

$$e^{-i(E/\hbar)t} G_0(\mathbf{x}, \mathbf{x}'; E) = \psi_0 \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r},$$

11 Apr p9

where $\psi_0 = e^{-i\mathbf{k}\cdot\mathbf{x} - i(E/\hbar)t}$.

14.4 Review

13 Apr p1

We have been considering the steady state response problem

$$\left[L_0 + \sigma \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}') e^{-i\omega t}.$$

The steady state response for outgoing waves (i.e., that which satisfies the boundary condition for scattering) is

$$u(\mathbf{x}, t) = e^{-i\omega t} G(\mathbf{x}, \mathbf{x}', \lambda = \omega^2 + i\varepsilon) \quad (14.6)$$

where G solves

eq14.g

$$[L_0 - \sigma\lambda]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}').$$

We want to get the response $\Phi(\mathbf{x}, t)$ for scattering from a plane wave. We only need to let $|\mathbf{x}'|$ go to infinity:

$$\lim_{|\mathbf{x}'| \rightarrow \infty} u(\mathbf{x}', t) = \frac{e^{ikr'}}{4\pi r'} \Phi(\mathbf{x}, t).$$

This gives scattering from a plane wave. Φ is the solution of

$$[L_0 + \sigma \frac{\partial^2}{\partial t^2}] \Phi(\mathbf{x}, t) = 0$$

which satisfies the boundary condition of scattering. For the case of steady state response we can write

$$\Phi(\mathbf{x}, t) = e^{-i\omega t} \Phi(\mathbf{x}, \omega)$$

where $\Phi(\mathbf{x}, \omega)$ solves the equation

$$[L_0 - \sigma\omega^2] \Phi(\mathbf{x}, \omega) = 0.$$

This equation satisfies the boundary condition of scattering:

13 Apr p2

$$\Phi(\mathbf{x}, \omega) = e^{i\mathbf{k} \cdot \mathbf{x}} + \Phi_s(\mathbf{x}),$$

where $\Phi_s(\mathbf{x})$ has only outgoing waves. Note that

$$k^2 = \frac{\omega^2 \sigma}{\tau} = \frac{\omega^2}{c^2} \text{ and } \mathbf{k} = k\hat{n},$$

where $\sigma = \lim_{r \rightarrow \infty} \sigma(r)$ and $\tau = \lim_{r \rightarrow \infty} \tau(r)$.

14.5 Spherical Symmetry Degeneracy

We now compare the mathematics of the plane wave solution Φ with that of the eigen function u . The eigenvalue equation can be written

$$[L_0 - \sigma\omega^2]u_\alpha(\mathbf{x}, \omega^2) = 0.$$

In this equation u_α is a positive frequency eigen function with degeneracy number α and eigen value ω^2 . The eigen functions can be obtained directly from the Green's function by using

$$\frac{1}{\pi} \text{Im } G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2 + i\epsilon) = \sum_{\alpha} u_{\alpha}(\mathbf{x}, \omega^2) u_{\alpha}^*(\mathbf{x}', \omega^2).$$

For the case of spherical symmetry we have

$$\begin{aligned} u_{\alpha}(\mathbf{x}, \omega^2) &= u_{lm}(\mathbf{x}, \omega^2) \\ &= Y_{lm}(\theta, \varphi) u_l(r, \omega^2) \end{aligned}$$

13 Apr p3

where $l = 0, 1, \dots$ and $m = -l, \dots, 0, \dots, l$. The eigenvalues ω^2 are continuous: $0 < \omega^2 < \infty$. Because of the degeneracy, a solution of the differential equation may be any linear combination of the degenerate eigen functions:

$$\Phi(\mathbf{x}, \omega) = \sum c_{\alpha} u_{\alpha},$$

where the c_{α} 's are arbitrary coefficients. This relates the eigen function to the plane wave scattering solution. In the next chapter we will see how the c_{α} 's are to be chosen.

14.6 Comparison of Classical and Quantum

pr:ClasMech1

Mathematically we have seen that classical mechanics and quantum mechanics are similar. Here we summarize the correspondences between their interpretations.

classical wave theory

u = wave amplitude

quantum particle theory

ψ = probability amplitude

u^2 = energy density	$ \psi ^2$ = probability density
ω_n = natural frequencies	E_n = energy eigenvalues
Normal modes:	Stationary states
$u_n(\mathbf{x})e^{-i\omega_n t} = u_n(\mathbf{x}, t)$	$\psi_n(\mathbf{x})e^{-i(E_n/\hbar)t} = \psi_n(\mathbf{x}, t)$
$L_0 u_n = \sigma \omega_n^2 u_n$	$H \psi_n(\mathbf{x}) = E_n \psi_n(\mathbf{x})$
L_0 is positive definite:	$H^+ = H$:
$\omega_n^2 > 0$	$E_n \in R$

The scattering problem is like the eigen value problem but we look at the region of continuous spectrum. For scattering, we require that $E_n > 0$. In this continuum case, look at the eigen values from:

13 Apr p4

$$(H - E)\psi_\alpha(\mathbf{x}, E) = 0, \quad (14.7)$$

where α labels wave functions with degenerate eigenvalues. Rather than a wave we have a beam of particles characterized by some energy E . The substitution from classical mechanics to quantum mechanics is as follows:

eq13.25

$$\omega^2 \sigma \rightarrow E, \quad \tau \rightarrow \frac{\hbar^2}{2m}, \quad k^2 \rightarrow \frac{E}{\hbar^2/2m} = \frac{2mE}{\hbar^2} = \left(\frac{p}{\hbar}\right)^2.$$

This last equation is the De Broglie relation.

pr:DeBr1

Now we want to look at the solution for quantum mechanical scattering using Green's functions. Suppose we have a beam of particles coming in. This incident free wave has the form $e^{i\mathbf{k}\cdot\mathbf{x} - iEt/\hbar}$ which solve the free space hamiltonian

13 Apr p5

$$H_0 = \frac{\hbar^2}{2m} \nabla^2.$$

We want the solution to equation 14.7 which corresponds to scattering. That is, we want the solution for

$$(H - E)\Phi(\mathbf{x}, E) = 0$$

which is of the form

$$\Phi(\mathbf{x}, E) = e^{-i\mathbf{k}\cdot\mathbf{x}} + \Phi_s$$

where Φ_s has only outgoing waves.

To solve this we look at the Green's function.

$$(H - E)G(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}') \text{ for } \text{Im}E > 0,$$

with the appropriate boundary conditions. We make the substitution

$$\psi(\mathbf{x}, t) = e^{-i(E/\hbar)t} G(\mathbf{x}, \mathbf{x}'; E + i\varepsilon).$$

This corresponds not to a beam of particles but rather to a source of particles.

13 Apr p6
Stuff omitted

14.7 Summary

1. The Green's function equation for the inhomogeneous Schrödinger equation is

$$(H - E)G(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}'),$$

where

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}).$$

2. The Green's function for a bound-state spectra in terms of eigen wave functions is

$$G(\mathbf{x}, \mathbf{x}'; E) = \sum_n \frac{\psi_n(\mathbf{x})\psi_n(\mathbf{x}')}{E_n - E}.$$

3. There is a close connection between classical and quantum mechanics which is discussed in section 14.6.

14.8 References

See your favorite quantum mechanics text.

Chapter 15

Scattering in 3-Dim

Chapter Goals:

- State the asymptotic form of the response function due to scattering from a localized potential.
- Derive the scattering amplitude for a far-field observer due to an incident plane wave.
- Derive the far-field form of the scattering amplitude.
- Define the differential cross section and write it in terms of the scattering amplitude.
- Derive and interpret the optical theorem.
- Derive the total cross section for scattering from a hard sphere in the high energy limit.
- Describe the scattering of sound waves from an oscillating sphere.

15 Apr p1

We have seen that the steady state case reduces the Green's function problem to the equation

$$[L_0 - \lambda\sigma]G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}')$$

with RBC, where the linear operator is given by

$$L_0 = -\nabla \cdot \tau(\mathbf{x})\nabla + V(\mathbf{x}).$$

In the spherically symmetric case we have $V(r)$, $\sigma(r)$, and $\tau(r)$. In chapter 11 we saw that the Green's function can be written as an expansion in terms of spherical harmonics,

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_{lm} Y_l^m(\theta, \varphi) G_l(r, r'; \lambda) Y_l^{m*}(\theta', \varphi').$$

In the last chapter we saw how to solve for scattering from a point source and scattering from a plane wave. We did this for a particular case in the problem set. This all had nothing to do with spherical

Only partly true.

symmetry. Now consider the case of spherical symmetry. From chapter 3 we know that the radial Green's function can be written

$$G_l(r, r'; \lambda) = -\frac{u_1^l(r_<, \lambda) u_2^l(r_>, \lambda)}{r^2 \tau(r) W(u_1^l, u_2^l)}. \quad (15.1)$$

eq14.0

The u 's solve the same radial eigenvalue equation

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \tau(r) \frac{d}{dr} \right) + \frac{\tau(r) l(l+1)}{r^2} + V(r) - \lambda \sigma(r) \right] u_{1,2}^l = 0, \quad (15.2)$$

eq14.1

but different boundary conditions. The eigenfunction u_1 satisfies the boundary condition

15 Apr p2

$$\frac{\partial}{\partial r} u_1^l - \kappa u_1^l = 0 \text{ for } r = a$$

and as $a \rightarrow 0$ we replace this with the boundary condition $u_1^l(r)$ finite at $r = 0$. The eigenfunction u_2 satisfies the boundary condition u_2 finite as $r \rightarrow \infty$.

By comparing equation 15.2 with previous one-dimensional equations we have encountered, we identify the second and third terms as an effective potential,

$$V_{\text{eff}} = \frac{\tau(r) l(l+1)}{r^2} + V(r).$$

pr:Veff1

In quantum mechanics we have

$$V_{\text{eff}} = \frac{\hbar^2 l(l+1)}{2mr^2} + V(r).$$

15.1 Angular Momentum

The above spherical harmonic expansion for the Green's function was obtained by solving the corresponding eigenfunction equation for the angular part,

$$\hbar^2 L_{\theta\varphi} Y_l^m = \hbar^2 l(l+1) Y_l^m.$$

The differential operator $L_{\theta\varphi}$ can be related to angular momentum by recalling that the square of the angular momentum operator satisfies the equation

$$L_{\text{op}}^2 Y_l^m = \hbar^2 l(l+1) Y_l^m.$$

Thus we identify $\hbar^2 L_{\theta\varphi}$ as L_{op}^2 , the square of the angular momentum operator.

$$\hbar^2 L_{\theta\varphi} \equiv L_{\text{op}}^2.$$

In the central potential problem of classical mechanics it was found that

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + V(r),$$

where

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

pr:bfL1

and

$$L^2 = \mathbf{L} \cdot \mathbf{L}.$$

In quantum mechanics the momentum operator is $p = (\hbar/i)\nabla$, so that

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} \rightarrow \mathbf{L}_{\text{op}} = \frac{\hbar}{i} \mathbf{x} \times \nabla$$

and thus

$$\begin{aligned} L_{\text{op}}^2 &= \left(\frac{\hbar}{i} \mathbf{x} \times \nabla \right) \cdot \left(\frac{\hbar}{i} \mathbf{x} \times \nabla \right) \\ &= \hbar^2 L_{\theta\varphi}. \end{aligned}$$

This gives the relation between angular momentum in classical mechanics and quantum mechanics.

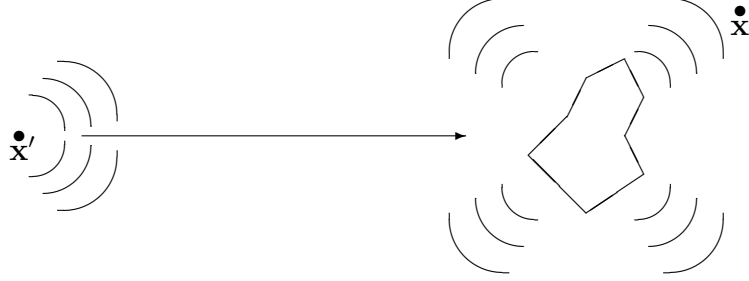


Figure 15.1: The schematic representation of a scattering experiment.

15.2 Far-Field Limit

pr:ExpScat1
fig14a

We now take the far field limit, in which $r \rightarrow \infty$, meaning the field is measured far from the obstacle. This situation is accurate for experimental scattering measurements and is shown in figure 15.1. We assume that in this $r \rightarrow \infty$ limit, we have $\sigma(r) \rightarrow \sigma$, $\tau(r) \rightarrow \tau$, and $rV(r) \rightarrow 0$. If instead the potential went as $V(r) = \gamma/r$, e.g., the Coulomb potential, then our analysis would change somewhat. We will also use the wave number $k = \sqrt{\sigma\lambda/\tau}$, where $\lambda = \omega^2 + i\epsilon$ classically, and $\lambda = E + i\epsilon$ for the quantum case. In classical mechanics we then have $k = \omega/c$ and in quantum mechanics we have $k = p/\hbar$.

15 Apr p3

Our incident wave is from a point source, but by taking the source-to-target separation r' big, we have a plane wave approximation. After making these approximations, equation 15.2 becomes

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} - k^2 \right] u_{l,2}^l = 0. \quad (15.3)$$

eq14.2

If we neglect the term $l(l+1)/r^2$, compared to k^2 we would need $kr \gg l$, which we don't want. Instead we keep this term in order to keep conventional solutions. In fact, it will prove easier to keep it, even though it may vanish faster than $V(r)$ as $r \rightarrow \infty$, and since we also have to consider l large, we don't want to kill it. In this limit the Green's function is proportional to a product of the u 's,

$$\lim_{r \rightarrow \infty} G_l(r, r'; \lambda) = A u_1^l(r', \lambda) u_2^l(r, \lambda). \quad (15.4)$$

We assume the the point source is not in the region where things are really happening, but far away. In this case $V(r') = 0$, $\sigma(r') = \sigma$, and $\tau(r') = \tau$, for large r' . Thus we are looking at the far field solution where the point source is outside the region of interaction. eq14.3
15 Apr p4

We already know the explicit asymptotic solution to the radial equation:

$$u_2^l(r_>) \approx h_l^{(1)}(kr_>) \quad \text{for } kr_> \gg 1, \quad (15.5)$$

$$u_1^l(r_<) \approx j_l^{(1)}(kr_<) + X_l h_l^{(1)}(kr_<) \quad \text{for } kr_< \gg 1. \quad (15.6)$$

X_l contains all the physics, which arises due to the boundary condition. eq14.5
In general, X_l must be evaluated numerically. For specific cases such as in the problem set, $V(r) = 0$ so equation 15.3 is valid everywhere, and thus we may obtain X_l explicitly. For our present situation we have assumed the far field approximation and an interaction-free source, for which the asymptotic form of the Green's function may be written in terms of equation 15.5 and 15.6 as

$$\lim_{r \rightarrow \infty} G_l(r, r'; \lambda) = A[j_l(kr') + X_l h_l^{(1)}(kr')] h_l^{(1)}(kr).$$

We have not yet specified $r > r'$, only that $r \gg 1$ and $r' \gg 1$.

To obtain the scattered wave at large r , we look at $G - G_0$. In particular we will evaluate $G_l - G_{l0}$. So we look at $G_0^l(r, r'; \lambda)$ for $r \gg r'$. Recall that G_{l0} has the form 15 Apr p5

$$G_{l0}(r, r'; \lambda) = \frac{ik}{\tau} j_l(kr_<) h_l^{(1)}(kr_>).$$

This is for the free problem; it solves all the way to the origin. Now take the difference.

$$G_l - G_{l0} = \left(A - \frac{ik}{\tau} \right) j_l(kr_<) h_l^{(1)}(kr_>) + A X_l h_l^{(1)}(kr_<) h_l^{(1)}(kr_>).$$

This equation assumes only that we are out of the range of of interaction. The term $h_l^{(1)}(kr_>)$ gives the discontinuity on dG/dr at $r = r'$. We now assert that A must equal ik/τ because the scattering wave $G_l - G_{l0}$ has to be nonsingular. Now look at the case $r > r'$: What does this mean?
15 Apr p6
pr:scatWv1

$$G_l - G_{l0} = \frac{ik}{\tau} X_l h_l^{(1)}(kr') h_l^{(1)}(kr).$$

All that is left to see is what the scattered wave looks like. We take kr to be large, as in the problem set.

To solve this equation not using Green's function, we first look for a solution of the homogeneous problem which at large distances gives scattered plus incident waves.

$$\Phi = \sum_{m=-\infty}^{\infty} c_m u^m(r, \varphi).$$

15 Apr p7

At large distances we have

$$\Phi \rightarrow e^{ixt} + \text{outgoing waves}.$$

18 Apr p1

18 Apr p2

15.3 Relation to the General Propagation Problem

We could instead consider the general problem of propagation, but at this time we are just considering the case of scattering, for which the source lies in a homogeneous region where $V(r) = 0$ and σ and τ are constant. The propagation problem is more general because it allows the source to be anywhere.

15.4 Simplification of Scattering Problem

For the scattering problem, we are considering a beam of particles from a distant ($r' \gg 1$) point source in a homogeneous medium incident on a target, which scatter and are detected by detectors far away ($r \gg 1$). This latter condition is called the far field condition. In this case we have seen that the problem can be simplified, and that we may explicitly calculate the scattered Green's function $G_{lS}(r, r'; \lambda)$:

pr:scGF1

$$G_{lS} = G_l(r, r', \lambda) - G_{l0}(r, r', \lambda) = \frac{ik}{c} X_l h_l^1(kr) h_l^1(kr'),$$

with $k = \sqrt{\lambda\sigma/\tau} = \sqrt{\lambda}/c$, where c is the speed. The value of X_l depends on k and is obtained from the behavior of u_1 at large $r = |\mathbf{x}|$. We already know that $u_1^l(r)$ must be of the form

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$$\lim_{r \rightarrow \infty} u_1^l(r) = j_l(kr) + X_l(k)h_l^1(kr),$$

since this is the asymptotic form of the solution of the differential equation. Thus scattering reduces to this form. All we need is X_l , which is obtained from the behavior of $u_1^l(r)$. In particular, we don't need to know anything about $u_2^l(r)$ if we are only interested in the scattering problem, because at large distances it cancels out.

What does this mean Physically? Isn't U_1^l for distances less than r_{source} ? Don't we need $r < r'$?

The large distance behavior of the function which satisfies the boundary condition at small distances is what determines the scattering solution. In this case r and r' are both large enough that we are in essentially a homogeneous region.

15.5 Scattering Amplitude

pr:scAmp1

Consider the special problem where $V = 0$, $\sigma = \text{const.}$, and $\tau = \text{const.}$, with the boundary condition

$$\frac{\partial u_1}{\partial r} + ku_1 = 0 \text{ for } r = a.$$

In the problem set we found X_l by satisfying this condition. The result was

$$X_l = -\frac{[kj_l'(ka) - \kappa j_l(ka)]}{[kh_l^{1'}(ka) - \kappa h_l^1(ka)]}.$$

This equation is valid for $r > a$.

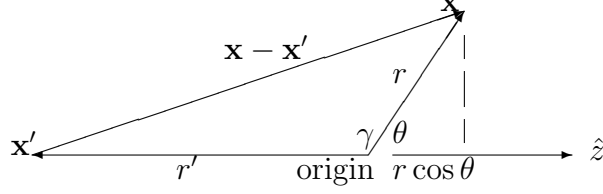
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Given X_l we can calculate the difference, $G_l - G_{l0}$ so we can calculate G_{IS} . Thus we can determine the scattered wave, which we now do.

We calculate the scattered piece by recalling the expansion in terms of spherical harmonics,

$$G_S = G - G_0 = \sum_{l,m} Y_l^m(\theta, \varphi) [G_l(r, r'; \lambda) - G_{l0}(r, r'; \lambda)] Y_l^{m*}(\theta', \varphi'). \quad (15.7)$$

We substitute into this the radial part of the the scattered Green's eq14.6

Figure 15.2: The geometry defining γ and θ .

function,

$$G(r, r'; \lambda) - G_0(r, r'; \lambda) = \sum_{l=0}^{\infty} \frac{ik}{c} X_l(k) h_l^{(1)}(kr) h_l^{(1)}(kr'), \quad (15.8)$$

eq14.7

and the spherical harmonics addition formula

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pr:addForm1

$$\begin{aligned} \sum_{m=-l}^l Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi') &= \frac{2l+1}{4\pi} P_l(\cos \gamma) \\ &= \frac{(-1)^l}{4\pi} (2l+1) P_l(\cos \theta) \end{aligned} \quad (15.9)$$

eq14.8

fig14b

where $\cos \gamma = \hat{x} \cdot \hat{x}'$. The geometry is shown in figure 15.2. The result of plugging equations 15.8 and 15.9 into equation 15.7 is

$$G_S = G - G_0 = \sum_{l=0}^{\infty} \frac{ik}{\tau} X_l(k) h_l^1(kr) h_l^1(kr') \frac{(-1)^l}{4\pi} P_l(\cos \theta) (2l+1).$$

15.6 Kinematics of Scattered Waves

We take the limit $kr \rightarrow \infty$ to get the far field behavior. In the asymptotic limit, the spherical Hankel function becomes

$$h^{(1)}(x) \xrightarrow{x \rightarrow \infty} -\frac{i}{x} (-i)^l e^{ix}.$$

Thus in the far field limit the scattered Green's function becomes

$$G - G_0 \rightarrow \frac{ik}{\tau} \frac{e^{ikr}}{kr} \frac{(-i)}{4\pi} \sum_{l=0}^{\infty} X_k h_l^1(kr') (i)^l (2l+1) P_l(\cos \theta).$$

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This is the case for a detector very far away. We can write also this as

$$G - G_0 = \frac{e^{ikr}}{r} \tilde{f}(\theta, r', k),$$

where

$$\tilde{f}(\theta, r', k) = \frac{1}{4\pi\tau} \sum_{l=0}^{\infty} (2l+1)(i)^l P_l(\cos\theta) X_l(k) h_l^1(kr'). \quad (15.10)$$

This is an independent proof that the scattered Green's function, $G - G_0$, is precisely an outgoing wave with amplitude \tilde{f} . eq14.9

The scattered part of the solution for the steady state problem is given by

$$u_s = e^{-i\omega t} (G - G_0).$$

The energy scattered per unit time per unit solid angle will be proportional to the energy per unit area, which is the energy flux u^2 . This in turn is proportional to the scattering amplitude \tilde{f} . Thus

$$\frac{dE}{dtd\Omega} \sim |\tilde{f}|^2.$$

We know the radial differential $ds = r^2 dr$ of the volume $dV = ds d\Omega$ for a spherical shell, so that we get

$$\frac{dE}{dtd\Omega} = \frac{dE}{dtds} \frac{ds}{dr}$$

Note that dimensionally we have $\frac{dE}{dtds} = \frac{1}{r^2}$ and $\frac{ds}{dr} = r^2$, so that $\frac{dE}{dtd\Omega}$ is dimensionless. Thus it is the $1/r$ term in the scattered spherical wave which assures conservation of energy. 18 Apr p7
pr:ConsE1

15.7 Plane Wave Scattering

We now look at scattering from a plane wave. Let $r' = |\mathbf{x}'|$ go to infinity. This gives us

$$h_l^{(1)}(kr') \xrightarrow{r' \rightarrow \infty} (-i)^{l+1} \frac{e^{|kr'|}}{kr'}.$$

In this limit equation 15.10 becomes

$$\tilde{f}(\theta, r', k) \rightarrow \frac{e^{|kr'|}}{4\pi\tau r'} f(\theta, k)$$

where

$$\frac{e^{kr'}}{r'} f(\theta, k) = -\frac{i}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) X_l. \quad (15.11)$$

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eq14fth
pr:frk1

$f(\theta, k)$ is called the scattering amplitude for a field observer from an incident plane wave. We can now compute the total wave for the far field limit with incident plane wave. It is

$$\begin{aligned} u &= e^{-i\omega t} G = e^{-i\omega t} [G - G_0 + G_0] \\ &= e^{-i\omega t} \left(-\frac{e^{ikr'}}{4\pi\tau r'} \right) \left(e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r} f \right). \end{aligned}$$

In this equation the term $e^{i\mathbf{k}\cdot\mathbf{x}}$ corresponds to a plane wave and the term $\frac{e^{ikr}}{r} f$ corresponds to an outgoing scattered wave. So

$$|f^2| \sim \frac{dE}{dt d\Omega}$$

This is a problem in the problem set.

15.8 Special Cases

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So far we have considered the case in which all the physics occurs within some region of space, outside of which we have essentially free space. We thus require that in the area exterior to the region, $V_0 = 0$, $\tau = \text{constant}$, and $\sigma = \text{constant}$. The source emits waves at \mathbf{x}' , and we want to find the wave amplitude at \mathbf{x} . Note that for the Coulomb potential, we have no free space, but we may instead establish a distance after which we may ignore the potential.

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15.8.1 Homogeneous Source; Inhomogeneous Observer

In this case \mathbf{x}' is in a region where $V(r) \approx 0$, and σ and τ are constant. We define u_0 to be the steady state solution to the point source problem without a scatterer present, i.e., u_0 is the free space solution.

$$u_0 = e^{-i\omega t} G_0,$$

where

$$G_0 = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi\tau|\mathbf{x}-\mathbf{x}'|}.$$

Further we define the scattered solution

$$u_s \equiv u - u_0.$$

To find u_s we use equation 15.1 to get the spherical wave expansion

$$G_{0l} = \frac{ik}{\tau} j_l(kr_<) h_l^{(1)}(kr_>).$$

Thus

$$u_s = e^{-i\omega t}(G - G_0).$$

For the case of a homogeneous source and an inhomogeneous observer $r_> = r'$, $r_< = r = 0$. We take

$$u_2^{(l)}(r') = h_l^{(1)}(kr').$$

Remember that r' is outside the region of scattering, so u_2^l solves the 20 Apr p4 free space equation, 15.3,

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} - k^2 \right] u_2 = 0, \quad (15.12)$$

where $k^2 = \lambda\sigma/\tau$ with the condition $u_2^{(l)}(r)$ finite as $r \rightarrow \infty$. The general eq14.10 solution to equation 15.12 is

$$u_2^{(l)}(r') = h_l^{(1)}(kr').$$

We still need to solve the full problem for u_1 with the total effective potential $V(r) \neq 0$.

15.8.2 Homogeneous Observer; Inhomogeneous Source

In this case the source point is in the interior region. We want to find 20 Apr p5 u for \mathbf{x} inside the medium, but we cannot use the u_s method as we did in case 1. The reason why it is not reasonable to separate u_0 and u_s in this case is because the source is still inside the scattering region.

We replace $r_> \rightarrow r$ and $r_< \rightarrow r'$ so that

$$u_2^{(l)}(r) \rightarrow h_l^{(1)}(kr)$$

and $u_1^{(l)}(r)$ satisfies the full potential problem. So once again we only need to solve for $u_1^{(l)}(r)$. The physics looks the same in case 1 and case 2, and the solutions in these two cases are reciprocal. This is a manifestation of Green's reciprocity principle. The case of a field inside due to a source outside looks like the case of a field outside due to a source inside.

15.8.3 Homogeneous Source; Homogeneous Observer

For this case both points are in exterior region. By explicitly taking $|\mathbf{x}| > |\mathbf{x}'|$ we make this a special case of the previous case. Thus we have $r_> \rightarrow r$ and $r_< \rightarrow r'$. Now both u_1 and u_2 satisfy the reduced ordinary differential equation

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} - k^2 \right] u_{1,2} = 0, \quad (15.13)$$

eq14rad

where u_1 satisfies the lower boundary condition and u_2 satisfies the upper boundary condition. As we have seen, the asymptotic solutions to this equation are

$$u_1^{(l)}(r') \rightarrow j_l(kr') + X_l h_l^{(1)}(kr'). \quad (15.14)$$

$$u_2^{(l)}(r) \rightarrow h_l^{(1)}(kr). \quad (15.15)$$

eq14.11,12

To obtain X_l we must solve

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$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \tau(r) \frac{d}{dr} \right) + \left(\frac{l(l+1)}{r^2} + V(r) \right) \tau(r) - \lambda \sigma \right] u_1 = 0,$$

and then take $r \gg 1$. We can then get $X_l(k)$ simply by comparing equations 15.14 and 15.15. The scattered wave is then

$$u_s = e^{-i\omega t} (G - G_0),$$

where

$$G_l - G_l^0 \rightarrow \frac{ik}{\tau} e^{-i\omega t} X_l h_l^{(1)}(kr) h_l^{(1)}(kr').$$

We see that the field at \mathbf{x} is due to source waves u_0 and scattered waves u_s .

Homogeneous Source and Observer, Far Field

For this case the source and the field point are out of the region of interaction. We take $r > r'$ and $r' \rightarrow \infty$.

For these values of r and r' we have $V = 0$ and τ and σ constant. In this case

$$e^{-i\omega t}G \rightarrow e^{-i\omega t}G_0 = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|-i\omega t}}{4\pi\tau|\mathbf{x}-\mathbf{x}'|} \quad (15.16)$$

(Eq.f)

$$G_0 = \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} (-1)^l P_l(\cos\theta) G_l^0 \quad (15.17)$$

where

$$G_l^0 = \frac{ik}{\tau} j_l(kr_{<}) h_l^{(1)}(kr_{>}) \quad (15.18)$$

(Eq.g)

$$u = e^{-i\omega t}G = e^{-i\omega t}G_0 + u_s \quad (15.19)$$

(Eq.h)

where

(Eq.i)

$$\begin{aligned} u_s &= e^{-i\omega t}(G - G_0) \\ &= \frac{ik}{\tau} e^{-i\omega t} \sum_{l=0}^{\infty} X_l \frac{(2l+1)(-1)^l}{4\pi} P_l(\cos\theta) h_l^{(1)}(kr) h_l^{(1)}(kr') \end{aligned}$$

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where for large r ,

$$u_1 \rightarrow j_l(kr) + X_l h_l^{(1)}(kr) \quad (15.20)$$

This is the large r behavior of the solution satisfying the small r boundary condition. (Eq.k)

15.8.4 Both Points in Interior Region

We put \mathbf{x} very far away, next to a detector. The assumption that \mathbf{x} lies in the vicinity of a detector implies $kr \gg 1$. This allows us to make the following simplification from case 2:

$$h_l^{(1)}(kr) \rightarrow (-i)^l \frac{(-i)}{kr} e^{ikr}. \quad (15.21)$$

Thus we can rewrite u_1 . We have

(Eq.m)

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$$u_s = e^{-i\omega t}(G - G_0) \quad (15.22)$$

and the simplification

(Eq.n)

$$G_0^l \rightarrow j_l(kr')h_l^{(1)}(kr). \quad (15.23)$$

(Eq.0)

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15.8.5 Summary

Here is a summary of the cases we have looked at

case 4 need to know u_1, u_2 everywhere
cases 1, 2 need to know u_1 everywhere
case 3 need to know u_1 at large r only

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We now look at two more special cases.

15.8.6 Far Field Observation

Make a large r expansion ($r \rightarrow \infty$):

$$h_l^{(1)}(kr) \rightarrow (-i)^l \left(\frac{-i}{kr}\right) e^{-ikr} \quad (15.24)$$

(EQ.l)

$$u = \frac{e^{-i(\omega t - k|\mathbf{x} - \mathbf{x}'|)}}{4\pi\tau|\mathbf{x} - \mathbf{x}'|} + \frac{e^{-i(\omega t - kr)}}{r} \tilde{f}(\theta, r', k). \quad (15.25)$$

(Eq.m)

The term $\frac{e^{-i(\omega t - kr)}}{r} \tilde{f}(\theta, r', k)$ is explicitly just the outgoing wave. We found

$$\tilde{f}(\theta, r', k) = \frac{1}{4\pi\tau} \sum_{l=0}^{\infty} (2l+1)(i)^l P_l(\cos\theta) X_l h_l^{(1)}(kr') \quad (15.26)$$

(Eq.n)

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The term $\tilde{f}(\theta, r', k)$ is called the scattering amplitude for a point source at r' . The flux of energy is proportional to \tilde{f}^2 .

15.8.7 Distant Source: $r' \rightarrow \infty$

Let the distance of the source go to infinity. Define

$$\mathbf{k} = k(-\hat{x}') \quad (15.27)$$

and in \tilde{f} , let $r' \rightarrow \infty$. This gives us

$$u \rightarrow \frac{e^{ikr'}}{4\pi r' \tau} \left[e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} + \frac{e^{-i(\omega t - kr)}}{r} f(\theta, k) \right] \quad (15.28)$$

We can then get

$$f = \frac{-i}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) X_l \quad (15.29)$$

This equation is seen in quantum mechanics. \tilde{f} is called the scattering amplitude at angle θ , and does not depend on φ due to symmetry. The basic idea is that plane waves come in, and a scattered wave goes out.

The wave number k comes from the incident plane wave.

(Eq.p)

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15.9 The Physical significance of X_l

Recall that X_l is determined by the large distance behavior of the solution which satisfies the short distance boundary condition. X_l is defined by

$$u_l^{(1)}(kr) \rightarrow j_l(kr) + X_l(k) h_l^{(1)}(kr). \quad (15.30)$$

This equation holds for large r with $V = 0$ and σ, τ constant. By using the identity

$$j_l(kr) = \frac{1}{2} \left(h_l^{(1)}(kr) + h_l^{(2)}(kr) \right),$$

we can rewrite equation 15.30 as

$$u_l^{(1)}(kr) \rightarrow \frac{1}{2} \left[h_l^{(2)}(kr) + (1 + 2X_l) h_l^{(1)}(kr) \right]. \quad (15.31)$$

We now define $\delta_l(k)$ by

$$1 + 2X_l = e^{2i\delta_l(k)},$$

eq14.21

We will prove that $\delta_l(k)$ is real. This definition allows us to rewrite equation 15.31 as

$$u_l^{(1)}(kr) \rightarrow \frac{1}{2} \left[h_l^{(2)}(kr) + e^{2i\delta_l(k)} h_l^{(1)}(kr) \right],$$

or

$$u_l^{(1)}(kr) = \frac{1}{2} e^{i\delta_l} \left[e^{-i\delta_l} h_l^{(2)} + e^{i\delta_l} h_l^{(1)} \right]. \quad (15.32)$$

eq14.22

The solution u_1^l satisfies a real differential equation. The boundary condition at $r \rightarrow 0$ gives real coefficients. Thus u_1^l is real up to an overall constant factor. This implies δ_l real. Another way of seeing this is to note that by the definition of $h_l^{(1)}$ and $h_l^{(2)}$ we have

$$h_l^{(2)}(kr) = \left[h_l^{(1)}(kr) \right]^*.$$

Thus the bracketed expression in equation 15.32 is an element plus its complex conjugate, which is therefore real. If $u_1^l(kr) \in R$, then $\delta_l(kr) \in R$.

Ask Baker
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We now look at the second term in equation 15.32 for far fields,

$$e^{i\delta_l} h_l^{(1)}(kr) \xrightarrow{r \rightarrow \infty} e^{i\delta_l(k)} \frac{-i}{kr} (-i)^l e^{ikr}.$$

Note that

$$(-i)^l = e^{-i\pi l/2}.$$

This gives

$$e^{i\delta_l} h_l^{(1)}(kr) = -\frac{i}{kr} e^{i(kr - \pi l/2 + \delta_l)}$$

So

$$u_1^l(kr) \sim \frac{1}{kr} \sin(kr - \pi l/2 + \delta_l(k)) \quad r \rightarrow \infty. \quad (15.33)$$

eq14.23

Thus $\delta_l(k)$ is the phase shift of the l th partial wave at wave number k . In the case that $V = 0$ we have

$$u_1^l(kr) \rightarrow u_{1,0}^l(kr).$$

If there is no potential, then we have

$$X_l(k) \rightarrow 0,$$

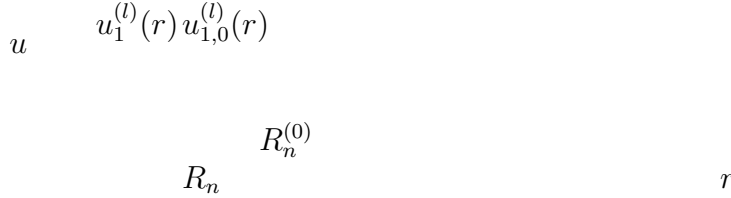


Figure 15.3: Phase shift due to potential.

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and by using the asymptotic expansion of j , we see that equation 15.30 becomes

$$u_{1,0}^l(r) \sim \frac{1}{kr} \sin \left(kr - \frac{\pi l}{2} \right). \quad (15.34)$$

Thus the phase shift $\delta_l(k)$ is zero if the potential is zero.

eq14.24

Consider the values of r for which the waves u_1 and $u_{1,0}$ are zero in the far field limit. For equation 15.33 and equation 15.34 respectively, the zeros occur when

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$$kR_n - \frac{\pi l}{2} + \delta_l = n\pi,$$

and

$$kR_n^0 - \frac{\pi l}{2} = n\pi.$$

By taking the difference of these equations we have

$$k(R_n - R_n^0) = -\delta_l(k). \quad (15.35)$$

Thus $\delta_l(k)$ gives the large distance difference of phase between solutions with interaction and without interaction. This situation is shown in figure 15.3. For the case shown in the figure, we have $R_n^0 > R_n$, which means $\delta_l > 0$. Note that turning on the interaction “pulls in” the scattered wave. Thus we identify two situations. $\delta_l > 0$ corresponds to an attractive potential, which pulls in the wave, while $\delta_l < 0$ corresponds to a repulsive potential, which pushes out the wave.

(eq14.25

fig14c

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We now verify this behavior by looking at the differential equation for the quantum mechanical case. We now turn to the quantum mechanical case. In this case we set $\tau(r) = \hbar^2/2m$ and $k^2(\mathbf{x}) = \frac{2mE}{\hbar^2}$ in the equation

$$\left[-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{V_{\text{eff}}^l(r)}{\tau(r)} - \frac{\lambda\sigma(r)}{\tau(r)} \right] u_1^l(r) = 0.$$

So for the radial equation with no interaction potential we have $\lambda\sigma/\tau = 2mE/\hbar^2$, while for the radial equation with an interaction potential we have $\lambda\sigma/\tau = 2m(E - V)/\hbar^2$. Thus the effect of the interaction is to change the wave number from

$$k_0^2 = \frac{2mE}{\hbar^2},$$

to an effective wave number

$$k^2(r) = \frac{2m(E - V)}{\hbar^2}. \quad (15.36)$$

eq14.26

Suppose we have an attractive potential, $V(r) > 0$. Then from equation 15.36 we see $k^2(r) > k_0^2$, which means momentum is increasing. Also, since $k^2(r) > 0$, increasing k^2 increases the curvature of u , which means the wavelength $\lambda(r)$ decreases and the kinetic energy increases. Thus the case $k^2(r) > k_0^2$ corresponds to an attractive potential pulling in a wave, which means $\delta_l(k) > 0$. The phase shift $\delta_l(k) > 0$ is a measure of how much the wave is pulled in. Note that this situation is essentially that of a wave equation for a wave moving through a region of variable index of refraction.

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fig14d

Now consider a repulsive potential with $l = 0$, as shown in figure 15.4. We have $V(r) = E$ for $r = r_0$, and $V(r) > E$ for $r > r_0$. In this latter case equation 15.36 indicates that $k^2(r) > 0$, which means the wave will be attenuated. Thus, as the wave penetrates the barrier, there will be exponential decay rather than propagation.

15.9.1 Calculating $\delta_l(k)$

From Griffies,
28 April, p2b

It is possible to calculate $\delta_l(k)$ directly from u_1^l without calculating $X_l(k)$ as an intermediate step. To do this, let $r \rightarrow \infty$ and then compare

$$V(r)$$

$$E$$

$$r_0$$

$$r$$

Figure 15.4: A repulsive potential.

this u_1^l with the general asymptotic form from equation 15.33, $u \sim \sin(kr - l\pi/2 + \delta_l(k))/r$. A different method for calculating $\delta_l(k)$ is presented in a later section.

15.10 Scattering from a Sphere

pr:ScSph1

We now look at the example of scattering from a sphere, which was already solved in the homework.

We have the boundary conditions

$$V = 0 \text{ at } r = a$$

$$\frac{\partial}{\partial r} u_1^l + \kappa u_1 = 0 \text{ at } r = a.$$

We found in problem set 2, that X_l for this problem is

25 Apr p7

$$X_l = \frac{[kj_l'(ka) - \kappa j_l(ka)]}{[kh_l^{(1)'}(ka) - \kappa h_l^{(1)}(ka)]} \quad (15.37)$$

by solving the radial equation.

Stuff missing

Now look at the long wavelength limit, which is also the low energy limit. In this case $ka \ll 1$ where $k = 2\pi/\lambda$. We know asymptotically that

$$j_l(ka) \sim (ka)^l,$$

and

$$h_l^{(1)}(ka) \sim \frac{1}{(ka)^{l+1}}.$$

Thus we have

$$X_l(k) \xrightarrow{ka \ll 1} (ka)^{2l+1} \left(\frac{l - \kappa a}{l + \kappa a} \right) \ll 1$$

since

$$(ka)^{2l+1} = \frac{(ka)^l}{(ka)^{-l-1}}.$$

25 Apr p8

Again, $k = \sqrt{2mE/\hbar^2}$. We now look at the phase shift for low energy scattering. We use the fact

$$X_l(k) \sim (ka)^{2l+1}$$

to write

$$\begin{aligned} 1 + 2X_l(k) &= e^{i2\delta(k)} \\ &= 1 + 2i\delta_l + \cdots. \end{aligned}$$

Thus we have

$$\delta_l(k) \sim (ka)^{2l+1}.$$

15.10.1 A Related Problem

25 Apr p9

We now turn to a related problem. Take an arbitrary potential, for example

$$V = V_0 e^{-r/a}.$$

fig14e

In this case the shape of V_{eff} is similar, except that it has a potential barrier for low values of r . V and V_{eff} for this example are shown in figure 15.5. The centrifugal barrier increases as l increases, that is, it gets steeper. Thus, as l increases, the scattering phase shift gets smaller and smaller since the centrifugal barrier gets steeper.

Recall that a represents the range of the potential and $2l + 1$ represents the effect of a potential barrier. We assert that in the long

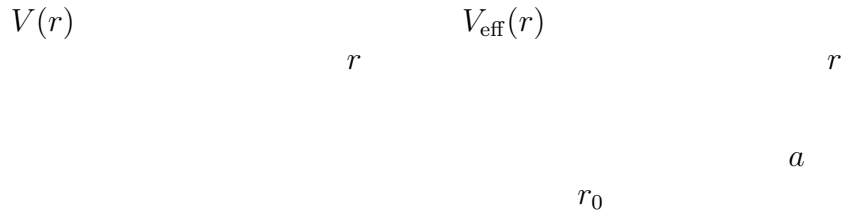


Figure 15.5: The potential V and V_{eff} for a particular example.

wave length limit, that is, low energy scattering, the phases shift goes generally as

$$\delta_l(k) \sim (ka)^{2l+1} \quad \text{for } ka \gg 1.$$

This is a great simplification for low energy scattering. It means that as long as $ka \ll 1$, we need only consider the first few l in the infinite series for the scattering amplitude $f(\theta)$. In particular, the dominant contribution will usually come from the $l = 0$ term. For the case $l = 0$, the radial equation is easier to solve, and $X_0(k)$ is easier to obtain. Thus the partial wave expansion is very useful in the long wavelength, or low energy, limit. This limit is the opposite of the geometrical or physical optics limit.

The low energy limit is useful, for example, in the study of the nuclear force, where the range of the potential is $a \sim 10^{-13}\text{cm}$, which gives $ka \ll 1$. Note that in the geometrical optics limit, $ka \gg 1$, it is also possible to sum the series accurately. The summation is difficult in the middle region, $ka \sim 1$. In this case many terms of the series must be retained.

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15.11 Calculation of Phase for a Hard Sphere

We use the “special case” from above. Take $\kappa \rightarrow \infty$ (very high elastic constant, very rigid media, a hard sphere). In this case $u \rightarrow 0$ when

$r = a$. Thus we get from equation 15.37

$$\begin{aligned} X_0(k) &= \frac{-\frac{\sin ka}{ka}}{-\frac{ie^{ika}}{ka}} \\ &= \frac{-ie^{ika} - e^{-ika}}{2ie^{ika}} \\ &= -\frac{1}{2}[1 - e^{-2ika}]. \end{aligned}$$

So

$$e^{2i\delta_0} = 1 - [1 - e^{-2ika}] = e^{-2ika},$$

and thus

$$\delta_0(k) = -ka. \quad (15.38)$$

In terms of quantum mechanics, this is like having

$$V(r) = \infty \text{ for } r < a,$$

$$V(r) = 0 \text{ for } r > a.$$

Outside we get the asymptotic solution form given in equation 15.33. For $l = 0$ and substituting equation 15.38, this becomes

$$u_1^{(0)} = \frac{1}{r} \sin(kr - ka).$$

By substituting this into equation 15.13 it is easy to verify that this is an exact solution for $u_1^{l=0}$. This is exactly what we would expect: a free space spherical wave which satisfies the boundary condition at $r = a$. The wave is pushed out by an amount ka . We thus see that $\delta_l(k)$ is determined by the boundary condition. This situation is shown in figure 15.6.

15.12 Experimental Measurement

We now look at the experimental consequences. Assume that we have solved for u_1 and know $X_l(k)$ and thus know $\delta_l(k)$. By writing the scattering amplitude from equation 15.11 in terms of the phase shift $\delta_l(k)$, we have

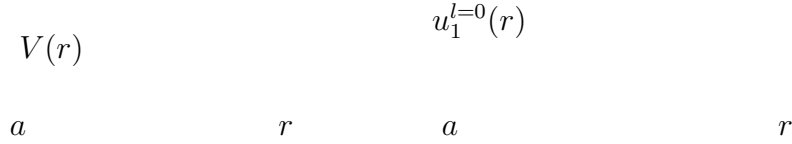


Figure 15.6: An infinite potential wall.

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) \frac{e^{2i\delta_l} - 1}{2i}.$$

To get δ_l for the solution for u_1 , we look at large r . Note that

$$\begin{aligned} \frac{e^{2i\delta_l} - 1}{2i} &= \frac{e^{i\delta_l} [e^{i\delta_l} - e^{-i\delta_l}]}{2i} \\ &= e^{i\delta_l} \sin \delta_l. \end{aligned}$$

So

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta). \quad (15.39)$$

eq14.55

15.12.1 Cross Section

pr:CrSec1

This scattering amplitude is the quantity from which we determine the energy or probability of the scattered wave. However, the scattering amplitude is not a directly measurable experimental quantity.

Recall our original configuration of a source, an obstacle, and a detector. The detector measures the number of particles intercepted per unit time, dN/dt . (It may also distinguish energy of the intercepted particle.) This number will be proportional to the solid angle covered by the detector and the incident flux of particles. If we denote the proportionality factor as $\sigma(\theta, \phi)$, then this relationship says that the rate at which particles are scattered into an element of solid angle is is $dN/dt = j_{\text{inc}} d\sigma = j_{\text{inc}} (d\sigma/d\Omega) d\Omega$. Note that an element of solid angle is related to an element of area by $r^2 d\Omega = dA$. The scattered current through area dA is then $dN/dt = j_{\text{inc}} (\sigma(\theta, \phi)/d\Omega) dA/r^2$. From this we

identify the scattered current density

$$\mathbf{j}_{\text{scat}} = j_{\text{inc}} \frac{\sigma(\theta, \phi)}{d\Omega} \frac{1}{r^2} \hat{r}. \quad (15.40)$$

eq14cs1

Now the quantum mechanical current density \mathbf{j} is defined in terms of the wave function:

$$\mathbf{j}(\mathbf{r}) = \text{Re} \left[\psi^\dagger \frac{\hbar}{im} \nabla \psi \right],$$

where in the far-field limit the boundary condition of scattering tells us that the wave function goes as

$$\psi_{\text{scat}} \xrightarrow{r \rightarrow \infty} N \left(e^{ikz} + \frac{e^{ikr}}{r} f(\theta, \phi) \right).$$

The wave function has an incident plane wave part and a scattered spherical wave part. The current density for the incident wave is then

$$\mathbf{j}_{\text{inc}} = |N|^2 \frac{\hbar k}{m} \hat{z} = j_{\text{inc}} \hat{z},$$

and the current density for the scattered wave is

$$\mathbf{j}_{\text{scat}} = |N|^2 \frac{\hbar k}{m} \frac{|f(\theta, \phi)|^2}{r^2} \hat{r} + \mathcal{O}(r^{-3}) \approx j_{\text{inc}} \frac{|f(\theta, \phi)|^2}{r^2} \hat{r}. \quad (15.41)$$

eq14cs2

By comparing equations 15.40 and 15.41, we identify the differential cross section as

$$\frac{d\sigma}{d\Omega} \equiv |f(\theta, k)|^2. \quad (15.42)$$

eq14.57

This relationship between cross section and scattering amplitude agrees with dimensional analysis. Note that the only dimensionful quantity appear in equation 15.39 for f is k :

$$\dim(f(\theta)) = \dim(k^{-1}) = \dim(\text{length}).$$

On the other hand, the dimension of the differential cross section is

$$\dim[d\sigma/d\Omega] = \dim[l^2/1] \text{ and } \dim[|f(\theta, k)|^2] = \dim[l^2].$$

Thus equation 15.42 is dimensionally valid. Note also that, because the differential cross section is an area per solid angle, it must be real and positive, which also agrees with $|f|^2$. The total cross section is

$$\sigma(k) = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega |f(\theta, k)|^2.$$

15.12.2 Notes on Cross Section

By using equation 15.39 we can calculate the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \sum_{l,l'=0}^{\infty} (2l+1)(2l'+1) e^{i\delta_l} \sin \delta_l e^{-i\delta_{l'}} \sin \delta_{l'} P_l(\cos \theta) P_{l'}(\cos \theta) \quad (15.43)$$

In this equation we get interference terms (cross terms). These interference terms prevent us from being able to think of the differential cross section as a sum of contributions from each partial wave individually. If we are measuring just σ , we can integrate equation 15.43 to get eq14.60
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$$\begin{aligned} \sigma &= \int d\Omega \frac{d\sigma}{d\Omega} \\ &= \int d\Omega \frac{1}{k^2} \sum_{l,l'=0}^{\infty} (2l+1)(2l'+1) e^{i\delta_l} \sin \delta_l e^{-i\delta_{l'}} \sin \delta_{l'} P_l(\cos \theta) P_{l'}(\cos \theta) \end{aligned} \quad (15.44)$$

We can simplify this by using the orthogonality of the Legendre polynomials: eq14.61

$$\int d\Omega P_l(\cos \theta) P_{l'}(\cos \theta) = \delta_{ll'} \frac{4\pi}{2l+1}.$$

In equation 15.44 the terms $e^{i\delta_l}$ cancel. So we now have

$$\begin{aligned} \sigma(k) &\equiv \int d\Omega |f(\theta, k)|^2 \\ &= \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l. \end{aligned} \quad (15.45)$$

From this we can conclude that eq14.64

$$\sigma = \sum_{l=0}^{\infty} \sigma_l,$$

where

$$\sigma_l = \frac{4\pi}{k^2} (2l+1) \sin^2 \delta_l. \quad (15.46)$$

Note that σ_l is the contribution of the total cross section of scattering from the $2l+1$ partial waves which have angular momentum l . There are no interference effects, which is because of spherical symmetry. eq14.65
27 Apr p7

Another way to think about this point is that the measuring apparatus has introduced an asymmetry in the field, and then we have interference effects in $d\sigma/d\Omega$. On the other hand, in the whole measurement of σ , there is still spherical symmetry, and thus no interference effects. A measurement of $\sigma(k)$ is much more crude than a measurement of $d\sigma/d\Omega$.

Because sine is bounded by one, the total cross section of the partial waves are also bounded:

$$\sigma_l \leq \sigma_l^{\max} = \frac{4\pi}{k^2}(2l+1),$$

or, by using $\lambda = 2\pi/k$,

$$\sigma_l^{\max} = 4\pi \left(\frac{\lambda}{2\pi} \right)^2 (2l+1). \quad (15.47)$$

eq14.67

Note that the reality of $\delta_l(k)$ puts a maximum value on the contribution $\sigma_l(k)$ of the l th partial wave on the total cross section.

pr:GeoLim1

15.12.3 Geometrical Limit

In the geometrical limit we have $ka \gg 1$, which is the long wavelength limit, $\lambda \gg a$. Recall that in this limit

$$\delta_l \sim (ka)^{2l+1}.$$

Thus the dominant contribution to the cross section will come from

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0 = \frac{4\pi}{k^2} (ka)^2 = 4\pi a^2. \quad (15.48)$$

From equation 15.47 we have

$$\sigma_0^{\max} \sim 4\pi \left(\frac{\lambda}{2\pi} \right)^2,$$

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and from equation 15.48 we have

$$\sigma_0 = 4\pi \lambda^2 \left(\frac{a}{\lambda} \right)^2.$$

Comparing these gives us $\sigma_0/\sigma_0^{\max} \ll 1$ for $a/\lambda \ll 1$, which is the fraction of the incident beam seen by an observer.

15.13 Optical Theorem

pr:OptThm1

We now take the imaginary part of equation 15.46:

$$\operatorname{Im} f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l P_l(\cos \theta).$$

In the case that $\theta = 0$ we get

$$\operatorname{Im} f(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l.$$

By comparing this with equation 15.45 we obtain

$$\operatorname{Im} f(0) = \sigma \frac{k}{4\pi}.$$

This is called the optical theorem. The meaning of this is that the imaginary part of the energy taken out of the forward beam goes into scattering. This principle is called unitarity or conservation of momentum. The quantity $\operatorname{Im} f(\theta)|_{\theta=0}$ represents the radiation of the intensity in the incident beam due to interference with the forward scattered beam. This is just conservation of energy: energy removed from the incident beam goes into the scattered wave.

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15.14 Conservation of Probability Interpretation:

15.14.1 Hard Sphere

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For the case of a hard sphere of radius a we found that

$$\delta_0 = -ka.$$

$\sigma k/4\pi$ as
a proportional-
ity factor

pr:HardSph1

In the case of $(ka) \ll 1$, only the lower terms of equation 15.39 matter. Exact scattering amplitude from a hard sphere $k = 0$? So for a sphere of radius a , we have

$$\delta_l \sim (ka)^{2l+1}.$$

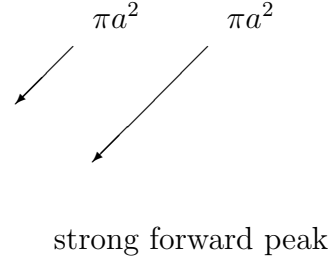


Figure 15.7: Scattering with a strong forward peak.

In the case that $k \rightarrow 0$, we get (noting that $e^{i\delta_l} \rightarrow 0$):

$$f(\theta) = \frac{i}{k} e^{i\delta_l} \sin \delta_0(k) P_0(\cos \theta) = \frac{i}{k} e^{i\delta_l} (-ka) = \frac{i}{k} (-ka) = -ia, \quad \text{as } k \rightarrow 0.$$

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Thus

$$\frac{d\sigma}{d\Omega} = a^2, \quad \text{as } k \rightarrow 0.$$

Note that the hard sphere differential cross section is spherically symmetric at low energy (that is, when $ka \ll 1$). In this case the total cross section is

$$\sigma \int \frac{d\sigma}{d\Omega} = 4\pi a^2.$$

For the geometrical optics limit, $ka \gg 1$, corresponding to short wavelength and high energy, we would expect $\sigma \sim \pi a^2$ since the sphere looks like a circle, but instead we get

$$\sigma \sim 2(\pi a^2).$$

The factor of two comes from contributions from all partial waves and has a strong forward peak. The situation has the geometry shown in figure 15.7. The figure is composed of a spherically symmetric part and a forward peak, which each contribute πa^2 to the total cross section σ .

fig14g

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15.15 Radiation of Sound Waves

pr:soundWv1

We consider a non-viscous medium characterized by a sound velocity v . In this medium is a hard sphere oscillating about the origin along

the z -axis. The motion of the center of the sphere is given by

$$\mathbf{x}_c = \varepsilon a e^{-i\omega t} \hat{z},$$

where $\varepsilon \ll 1$, and the velocity of the center of the sphere is then given by

$$\mathbf{v}_c = -i\omega a e^{-i\omega t} \hat{z}.$$

Note that the normal component of the velocity at the surface of the sphere is

$$\hat{n} \cdot \mathbf{v}_{\text{sphere}} = -i\varepsilon\omega a e^{-i\omega t} \cos \theta. \quad (15.49)$$

where we have used $\hat{n} \cdot \hat{z} = \cos \theta$ with θ measures from the \hat{z} -axis. The minus sign appears because we choose n to point into the sphere. For the velocity of the fluid outside of the hard sphere we have eq14.88

$$\mathbf{v}_{\text{fluid}} = \nabla \Phi,$$

where Φ is the velocity potential. Thus near the surface of the sphere we have, up to first order in ε ,

$$\hat{n} \cdot \mathbf{v}|_{r=a} = \frac{\partial \Phi}{\partial r} \Big|_{r=a}.$$

We want to find the velocity potential where the velocity potential satisfies the equation

$$\left[\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Phi(\mathbf{x}, t) = 0, \quad r > a,$$

with the hard sphere boundary condition that the fluid and the sphere move at the same radial velocity near the surface of the sphere,

$$\hat{n} \cdot \mathbf{v}_{\text{sphere}} = \hat{n} \cdot \mathbf{v}_{\text{fluid}}.$$

The velocity of the fluid is then given by (using equation 15.49)

$$\frac{-\partial \Phi}{\partial r} \Big|_{r=a} = -i\varepsilon\omega a e^{i\omega t} \cos \theta. \quad (15.50)$$

eq14.93

15.15.1 Steady State Solution

pr:sss4

The steady state solution is of the form

$$\Phi(\mathbf{x}, t) = e^{-i\omega t} \Phi(\mathbf{x}, \omega),$$

where $\Phi(\mathbf{x}, \omega)$ satisfies

$$[\nabla^2 + l^2] \Phi(\mathbf{x}, \omega), \quad r > a,$$

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with the outgoing wave boundary condition (from equation 15.50)

$$\frac{\partial \Phi}{\partial r} = -i\varepsilon\omega a \cos(\theta), \quad r = a.$$

Our boundary condition is of the form

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = g(\theta, \varphi),$$

where for our specific case

$$g(\theta, \varphi) = -i\varepsilon\omega a \cos \theta. \quad (15.51)$$

eq14.98

We want to solve the steady state equation subject to the boundary condition. A more general form of the boundary condition is

$$\frac{\partial \Phi}{\partial r} + \kappa \Phi = g(\theta, \varphi). \quad (15.52)$$

eq14.99

We write

$$[-\nabla^2 - k^2] G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x}' - \mathbf{x})/c^2,$$

where $k^2 = \lambda/c^2$. This is the standard form of the Green's function in the case that $\tau = c^2$, $L_0 = \tau \nabla^2$, and $|\mathbf{x}'| |\mathbf{x}| > 0$. The solution of this equation, which we found previously, is

$$c^2 G(\mathbf{x}, \mathbf{x}'; \lambda) = ik \sum_{lm} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \frac{z_l(kr_{<}) h_l^{(1)}(kr_{>})}{\kappa h_l^{(1)}(ka) - k h_l^{(1)'}(ka)}. \quad (15.53)$$

(eq14.101

The general solution is given by a superposition of the Green's function solution for source points on the surface of the sphere,

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$$\Phi(\mathbf{x}) = c^2 \int_{\mathbf{x}' \in S} G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2 + i\varepsilon) g(\theta, \varphi) a^2 d\Omega'. \quad (15.54)$$

By setting $\lambda = \omega^2 + i\varepsilon$, we have automatically incorporated the outgoing wave condition. Physically, $g(\theta, \varphi) a^2 d\Omega'$ is the strength of the disturbance.

We use equation 15.53 with $r_> = 0$ and $r_< = a$. Thus z_l is

$$\begin{aligned} z_l(kr) &= [\kappa h_l^{(1)}(ka) - k h_l^{(1)'}(ka)] j_l(kr) \\ &\quad - [\kappa j_l'(ka) - j_l'(ka)] h_l^{(1)}(kr) \\ &= -kW(j_l(ka) h_l^{(1)}(ka)). \end{aligned}$$

Recall that we have evaluated this Wronskian before, and plugging in the result gives

$$z_l(ka) = -k \frac{i}{(ka)^2} = -\frac{i}{ka^2} \quad (15.55)$$

By combining equations 15.55 and 15.53 into equation 15.54, we obtain

$$\begin{aligned} \Phi(\mathbf{x}) &= -a^2 \left(\frac{i}{ka^2} \right) (ik) \sum_{lm} \frac{Y_{lm}(\theta, \varphi) h_l^{(1)}(kr)}{\kappa h_l^{(1)}(ka) - k h_l^{(1)'}(ka)} \int d\Omega' Y_{lm}^*(\theta', \varphi') g(\theta', \varphi') \\ &= \sum_{lm} \frac{Y_{lm}(\theta, \varphi) h_l^{(1)}(kr)}{\kappa h_l^{(1)}(ka) - k h_l^{(1)'}(ka)} g_{lm}. \end{aligned} \quad (15.56)$$

This is called the multipole expansion, where we also have defined

$$g_{lm} \equiv \int d\Omega' Y_{lm}^*(\theta', \varphi') g(\theta', \varphi'). \quad (15.57)$$

g_{lm} is the (l, m) th multipole moment of $g(\theta, \varphi)$.

15.15.2 Far Field Behavior

At distances far from the origin ($r \rightarrow \infty$) the spherical Hankel functions can be approximated by

$$h_l^{(1)}(k, r) = -\frac{i}{kr} (-i)^l e^{ilr}, \quad r \gg 1.$$

In this limit the velocity potential can be written

$$\Phi(\mathbf{x}) = \frac{e^{ikr}}{r} f(\theta, \varphi), \quad r \rightarrow \infty,$$

where the amplitude factor f is given by

$$f(\theta, \varphi) = -\frac{i}{k} \sum_{l,m} \frac{g_{l,m} Y_l^m(\theta, \varphi) (-i)^l}{\kappa h_l^{(1)}(k, a) - k h_l^{(1)'}(k, a)}.$$

This amplitude may be further decomposed into components of particular l and m :

$$f = \sum_{l,m} f_{l,m} Y_l^m(\theta, \varphi),$$

where

$$f_{l,m} = \frac{(-i)^{l+1}}{k} \frac{g_{l,m}}{\kappa h_l^{(1)}(k, a) - k h_l^{(1)'}(k, a)}.$$

The interpretation of the l 's is

$l = 1$	dipole radiation	$m = 0, \pm 1$
$l = 2$	quadrupole radiation	$m = 0, \pm 1, \pm 2$
$l = 3$	octopole radiation	$m = 0, \pm 1, \pm 2, \pm 3$

For the case $l = 1$, the 3 possible m 's correspond to different polarizations.

2 May p3

15.15.3 Special Case

We now return to the specific case of the general boundary condition, equation 15.52, which applies to a hard sphere executing small oscillations. In this case the hard surface implies $\kappa = 0$ and the oscillatory motion implies that g is given by equation 15.51, which can be rewritten in terms of the spherical harmonic Y_1^0 :

$$g(\theta, \varphi) = -i\omega a \varepsilon \sqrt{\frac{3}{4\pi}} Y_1^0(\theta, \varphi).$$

By plugging this into equation 15.57, we have obtained the (l, m) components of $g(\theta, \phi)$,

$$\begin{aligned} g_{l,m} &= \int d\Omega Y_l^m(\theta, \phi) g(\theta, \phi) \\ &= \int d\Omega (-i\omega a \varepsilon) Y_l^m(\theta, \phi) \sqrt{\frac{3}{4\pi}} Y_1^0(\theta, \varphi) \\ &= -i\omega a \varepsilon \sqrt{\frac{3}{4\pi}} \delta_{m0} \delta_{l1}. \end{aligned}$$

This shows that the oscillating sphere only excites the Y_l^0 mode.

$$\begin{aligned} f &= -\frac{1}{k} \frac{-i\omega a t}{k - kh_l^{(1)'}(ka)} \sqrt{\frac{3}{4\pi}} Y_l^0(\theta, \varphi) \\ &= \frac{-ia\varepsilon\omega \cos \theta}{k^2 h_l^{(1)'}(ka)}. \end{aligned}$$

Thus we have pure dipole radiation for this type of oscillation. This final equation gives the radiation and shows the dependence on k .

omitted qm
stuff
2 May p4

15.15.4 Energy Flux

Consider a sound wave with velocity

$$\mathbf{v} = -\nabla \Phi(\mathbf{x}, t),$$

where in the far field limit the velocity potential is

$$\Phi(\mathbf{x}, t) \rightarrow \frac{e^{ikr}}{r} f(\theta, \varphi) e^{-i\omega t}, \quad \mathbf{x} \rightarrow \infty.$$

We now obtain the rate dE/dt at which energy flows through a surface. This is given by the energy flux through the surface,

$$\frac{dE}{dt} \equiv \int d\mathbf{s} \cdot \mathbf{j}_E,$$

where \mathbf{j}_E is the energy flux vector. For sound waves the energy flux vector can be expressed as a product of velocity and pressure,

$$\mathbf{j}_E = \mathbf{v}p. \quad (15.58)$$

This can be intuited as follows. The first law of thermodynamics says *eq14jvp* that for an ideal fluid undergoing a reversible isentropic process, the *FW p299* change in internal energy dE matches the work done on the element, $-pdV$. The total energy flowing out of through the surface S is

$$\int_S d\mathbf{s} \cdot \mathbf{j}_E = \frac{dE_s}{dt} = \int_S ds \frac{dE}{dt} = \int_S p dV = \int_S d\mathbf{s} \cdot \left(p \frac{d\mathbf{r}}{dt} \right).$$

By comparing integrands we obtain equation 15.58, as desired. Note that $\mathbf{j}_E = \mathbf{v}p$ has the correct dimensions for flux — that is, velocity times pressure gives the correct dimensions for energy.

2 May p5

The velocity and pressure are defined in terms of the velocity potential and density,

$$\mathbf{x} = -\nabla\Phi, \quad p = \rho \frac{\partial\Phi}{\partial t}.$$

We now look at the real parts of the velocity and the pressure for the steady state solution,

$$\text{Re } \mathbf{v} = \frac{1}{2} \mathbf{v} e^{-i\omega t} + \mathbf{v}^* e^{i\omega t},$$

$$\text{Re } p = \frac{1}{2} (p e^{-i\omega t} + p^* e^{i\omega t}).$$

ask Baker The flux is then about this.

$$\begin{aligned} \mathbf{j} &= \text{Re } v_R \text{Re } p_R \\ &= \frac{1}{4} (\mathbf{v} p^* + \mathbf{v}^* p) + e^{-2i\omega t} \mathbf{v} p + e^{2i\omega t} \mathbf{v}^* p^*. \end{aligned}$$

The time averaged flux is then

$$\langle \mathbf{j} \rangle = \frac{1}{4} (\mathbf{v} p^* + \mathbf{v}^* p) = \frac{1}{2} \text{Re } (\mathbf{x} p^*),$$

where we have used

$$\langle e^{-2i\omega t} \rangle + \langle e^{+2i\omega t} \rangle = 0.$$

The angled brackets represents the average over time. Note that \mathbf{x} and p^* are still complex, but with their time dependence factored out. To obtain p we use

$$p(t) = e^{-i\omega t} p = -\rho\omega\Phi e^{-i\omega t},$$

from which we obtain

$$p = -\rho\omega\Phi.$$

Thus the time averaged flux is

$$\langle \mathbf{j}_E \rangle = \frac{1}{2} \text{Re}(-\nabla\Phi)(+i\omega\epsilon)\Phi^*. \quad (15.59)$$

The radial derivative of Φ is

eq14.149

$$-\hat{r}\nabla\Phi = -\frac{\partial\Phi}{\partial r} = -ik\Phi.$$

Thus in this case the time averaged energy rate is

$$\left\langle \frac{dE}{dt} \right\rangle = r^2 d\Omega \hat{r} \cdot \langle \mathbf{j}_E \rangle = r^2 d\Omega \frac{|f|^2}{r^2} \omega \rho k.$$

Therefore

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} \rho k \omega |f|^2, \quad r \gg 1. \quad (15.60)$$

eq14.152

Plane Wave Approximation

pr:PIWv2

Now suppose that instead of a spherical wave, we have a plane wave,

$$\Phi(\mathbf{x}, t) = e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}.$$

In this special case the velocity and pressure are given by

$$\mathbf{v} = -\nabla\Phi = i\mathbf{k}\Phi \quad r \gg 1,$$

$$p = \rho(-i\omega)\Phi.$$

Using equation 15.59, the energy flux is

$$\mathbf{j} = \frac{1}{2} \mathbf{k} \omega = \frac{1}{2} k \omega \rho.$$

The power radiated through the area element dA is then

2 May p8

$$\frac{dE_A}{dt} = \int_{dA} d\mathbf{s} \cdot \mathbf{j} = \frac{\rho k \omega}{2} dA,$$

and we have

$$\frac{1}{dA} \frac{dE_A}{dt} \equiv \text{Incident flux} = \frac{\rho k \omega}{2}. \quad (15.61)$$

(eq14.158

15.15.5 Scattering From Plane Waves

The far field response to scattering from an incident plane wave is

$$\Phi(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + f \frac{e^{ikr}}{r}.$$

Note that in the limit $r \rightarrow \infty$, the scattered wave Φ_S is $f e^{ikr}/r$. So

$$\frac{dE_s}{dt d\Omega} = \frac{1}{2} \rho k \omega |f|^2.$$

2 May p5

By definition, the differential cross section is given by the amount of energy per unit solid angle per unit time divided by the incident energy flux,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &\equiv \frac{\frac{dE_s}{dt d\Omega}}{\text{Incident flux}} \\ &= \frac{\frac{1}{2} \rho k \omega |f|^2}{\frac{1}{2} \rho k \omega} = |f|^2. \end{aligned}$$

In the second equality we have used equation 15.60 and 15.61. This duplicates our earlier result, equation 15.42.

If we are just interested in the radiated wave and not the incident flux, the angular distribution of power is

$$\frac{dP}{d\Omega} = \frac{dE}{dt d\Omega} = \frac{1}{2} k \rho \omega |f|^2.$$

Now expand f in terms of spherical harmonics,

$$f = \sum_{l,m} Y_{l,m} f_{l,m}.$$

The radiated differential power can then be written

$$\frac{dP}{d\Omega} = \frac{1}{2}k\rho\omega|f|^2 = \frac{1}{2}k\rho\omega \left(\sum_{l,m} Y_l^m f_{l,m} \right) \left(\sum_{l',m'} Y_{l'}^{m'*} f_{l',m'}^* \right),$$

where we have interference terms. The total power is

2 May p6

$$P = \int d\Omega \frac{dP}{d\Omega} = \sum_{l,m} |f_{l,m}|^2.$$

In this case there is no interference. This is the analogue for sound wave of the differential cross section we studied earlier. For the case of a sphere

$$f_{l,0} \neq 0.$$

4 May p1

15.15.6 Spherical Symmetry

We now consider the situation where the properties of the medium surrounding the fluid exhibit spherical symmetry. In this case the scattering amplitude can be expanded in terms of spherical harmonics,

$$f(\theta, \varphi) = \sum_{l,m} f_{l,m} Y_l^m(\theta, \varphi).$$

This is called the multipole expansion. The term $f_{l,m}$ corresponds to the mode of angular momentum radiation. Spherical symmetry here means that the dynamic terms are spherically symmetric: $\sigma(r)$, $\tau(r)$, and $V(r)$. However, any initial condition or disturbance, such as g , may have asymmetry. We now look at the external distance problem.

$$g_{l,m} = \int d\Omega Y_l^{m*}(\theta, \varphi) g(\theta, \varphi).$$

For the general boundary condition the scattering amplitude is related to g by

$$f_{l,m} = \frac{(-i)^{l+1}}{k} \frac{g_{l,m}}{\kappa h_l^{(1)}(ka) - k h_l^{(1)'}(ka)}.$$

For our case of small oscillations of a hard sphere, we have $\kappa = 0$ and 4 May p2

$$g_{l,m} = -\delta_{l,1}\delta_{m,0}\sqrt{\frac{3}{4\pi}}i\varepsilon a\omega.$$

In this case the scattering amplitude becomes

$$f(\theta, \varphi) = \sum_{l,m} F_{lm} Y_l^m(\theta, \varphi) = \frac{-ia\varepsilon\omega}{k^2 h_{l=1}^{(1)}(ka)} \cos \theta = \frac{-ia\varepsilon c \cos \theta}{k h_1^{(1)'}(ka)}.$$

Thus the differential power radiated is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2}\rho k^2 \left| \frac{a\varepsilon c}{k} \frac{\cos \theta}{h_1^{(1)'}(ka)} \right| \\ &= \frac{1}{2} \frac{\rho a^2 \varepsilon^2 c^2}{|h_1^{(1)'}(ka)|^2} \cos^2 \theta. \end{aligned}$$

Notice that this $\cos^2 \theta$ dependence is opposite that of dipole radiation, which goes like $\sin^2 \theta$. The total power radiated is in general given by

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{1}{2}\rho c k^2 \sum_{l,m} |f_{l,m}|^2.$$

4 May p3

Note that there are no interference terms. It is simply a sum of power from each partial wave.

15.16 Summary

1. The asymptotic form of the response function is

$$\lim_{r \rightarrow \infty} u_1^l(r) = j_l(kr) + X_l(k) h_l^1(kr).$$

2. The scattering amplitude for a far-field observer due to an incident plane wave is

$$f(\theta, k) = -\frac{i}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) X_l.$$

3. The phase shift $\delta_l(k)$ is defined by the relation

$$1 + 2X_l = e^{2i\delta_l(k)},$$

which results in a scattered wave solution of the form

$$u_1^l(kr) \sim \frac{1}{kr} \sin(kr - \pi l/2 + \delta_l(k)) \quad r \rightarrow \infty,$$

where $\delta_l(k)$ appears as a simple shift in the phase of the sine wave.

4. The scattering amplitude is given by

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta).$$

5. The differential cross section represents the effective area of the scatterer for those particle which are deflected into the solid angle $d\Omega$, and can be written in terms of the scattering amplitude as

$$\frac{d\sigma}{d\Omega} \equiv |f(\theta, k)|^2.$$

6. The optical theorem is

$$\text{Im} f(\theta)|_{\theta=0} = \sigma \frac{k}{4\pi}.$$

It relates forward wave to the scattered wave.

7. The total cross section for scattering from a hard sphere in the high energy limit is

$$\sigma \sim 2(\pi a^2).$$

15.17 References

See any *old* nuclear or high energy physics text, such as [Perkins87].

Chapter 16

Heat Conduction in 3D

Chapter Goals:

- State the general response to the time-dependent inhomogeneous heat equation.
- Describe the physical significance of the boundary condition.
- Derive the temperature exterior to a fixed temperature circle.

16.1 General Boundary Value Problem

We saw in an earlier chapter that the heat equation is

$$\left[L_0 + \rho c_p(\mathbf{x}) \frac{\partial}{\partial t} \right] T(\mathbf{x}, t) = \rho q(\mathbf{x}, t)$$

for \mathbf{x} in R , with the linear operator

$$L_0 = -\nabla \kappa_T(\mathbf{x}) \nabla.$$

For the time dependent problem need both an initial condition and a boundary condition to determine a unique solution. The initial condition is

$$T(\mathbf{x}, t) = T_0(\mathbf{x}) \quad \text{for } t = 0.$$

For our boundary condition we take the radiation condition,

$$\kappa_T \hat{n} \cdot \nabla T = \alpha [T_{\text{ext}}(s, t) - T(\mathbf{x}, t)] \quad \text{for } \mathbf{x} \text{ on } s.$$

Recall the the radiation condition came from the equilibrium condition for radiation conduction balance. As an example of this sort of problem, consider the boundary to be the surface of the earth. In the evening time the temperature of the surface is determined by radiation. This is a faster method of transfer than heat conduction. The above radiation condition says that there exists a radiation conduction balance. Note that when we consider convection, we must keep the velocity dependent term $\mathbf{x} \cdot \nabla$ and the problem becomes non-linear. In this context \mathbf{v} is the motion of the medium due to convection.

The solution in terms of the Green's function is given by the principle of superposition

$$\begin{aligned} T(\mathbf{x}, t) = & \int_0^t dt' \int_R d\mathbf{x}' G(\mathbf{x}, t; \mathbf{x}', t') \rho(\mathbf{x}') \dot{q}(\mathbf{x}', t') \\ & + \int_0^t dt' \int_{\mathbf{x} \in S} ds' G(\mathbf{x}, t; \mathbf{x}', t') \alpha T_{\text{ext}}(s', t') \\ & + \int_R G(\mathbf{x}, t; \mathbf{x}', 0) \rho(\mathbf{x}') c_p(\mathbf{x}') T_0(\mathbf{x}'). \end{aligned}$$

eq15.0
4 May p4

The integral containing $\rho(\mathbf{x}') \dot{q}(\mathbf{x}', t')$ represents contributions due to volume sources; the integral containing $\alpha T_{\text{ext}}(s', t')$ represents contributions due to surface sources; and the integral containing $\rho(\mathbf{x}') c_p(\mathbf{x}') T_0(\mathbf{x}')$ represents contributions due to the initial conditions. The integrations over time and space can be done in either order, which ever is easiest. The Green's function is given by

$$G(\mathbf{x}, t; \mathbf{x}', t') = \int_L \frac{ds}{2\pi i} e^{s(t-t')} G(\mathbf{x}, \mathbf{x}'; \lambda = -s), \quad (16.1)$$

eq15.1

where L is the upward directed line along any constant $\text{Re } s > 0$. This choice of contour is necessary since L_0 is positive definite, which means that all the singularities of $G(\mathbf{x}, \mathbf{x}'; \lambda = -s)$ lie on the negative real s axis. This integral, which gives the inverse Laplace transform, is sometimes called the Bromwich integral. The Laplace space Green's function satisfies the differential equation

pr:Brom1

$$[L_0 - \lambda \rho c_p] G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}') \quad \mathbf{x}, \mathbf{x}' \in R,$$

and the boundary condition

$$[\kappa_T \hat{n} \cdot \nabla + \alpha]G(\mathbf{x}, \mathbf{x}', \lambda) = 0 \quad \mathbf{x} \in R, \mathbf{x}' \in S.$$

If the dynamical variables $c_p(\mathbf{x})$, $\rho(\mathbf{x})$, and $\kappa_T(\mathbf{x})$ are spherically symmetric, then the Green's function can be written as bilinear product of spherical harmonics,

$$G(\mathbf{x}, \mathbf{x}'; \lambda) = \sum Y_l^m(\theta, \varphi) G_l(r, r'; \lambda) Y_l^{m*}(\theta', \varphi').$$

By plugging this into equation 16.1, we obtain

$$G(\mathbf{x}, t, \mathbf{x}', t') = \sum_{l,m} Y_l^m(\theta, \varphi) G_l(r, t; r', t') Y_l^{m*}(\theta', \varphi')$$

where

$$G(r, t; r', t') = \int_L \frac{ds}{2\pi i} e^{s(t-t')} G_l(r, r'; \lambda = -s).$$

7 May p1

16.2 Time Dependent Problem

We now consider the case in which the temperature is initially zero, and the volume and surface sources undergo harmonic time dependence:

$$\begin{aligned} T_0(\mathbf{x}, t) &= 0 \\ \rho \dot{q}(\mathbf{x}, t) &= \rho \dot{q}(\mathbf{x}) e^{-i\omega t} \\ \alpha T_{\text{ext}}(s', t) &= \alpha T_{\text{ext}}(s') e^{-i\omega t}. \end{aligned}$$

We want to find $T(\mathbf{x}, t)$ for $t > 0$. Note that if $T_0(\mathbf{x}) \neq 0$ instead, then in the following analysis we would also evaluate the third integral in equation 16.1. For the conditions stated above, the temperature response is

$$\begin{aligned} T(\mathbf{x}, t) &= \int_0^t dt' \int_R d\mathbf{x}' G(\mathbf{x}, t; \mathbf{x}', t') \rho(\mathbf{x}') \dot{q}(\mathbf{x}') e^{-i\omega t'} \\ &\quad + \int_0^t dt' \int_{\mathbf{x} \in S} ds' G(\mathbf{x}, t; \mathbf{x}', t') \alpha T_{\text{ext}}(s') e^{-i\omega t'}. \end{aligned}$$

7 May p2

We are looking for the complete time response of the temperature rather than the steady state response. The time integration is of the form

$$\begin{aligned}
 \int_0^t dt G(\mathbf{x}, t; \mathbf{x}', t') e^{-i\omega t'} &= \int_L \frac{ds}{2\pi i} e^{st} G(\mathbf{x}, \mathbf{x}'; \lambda = -s) \int_0^t e^{-st' - i\omega t'} dt' \\
 &= \int_L \frac{ds}{2\pi i} e^{st} G(\mathbf{x}, \mathbf{x}'; \lambda = -s) \frac{1 - e^{-(s+i\omega)t}}{s + i\omega} \\
 &= \int_L \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}'; \lambda = -s)}{s + i\omega} [e^{st} - e^{-i\omega t}]. \quad (16.2)
 \end{aligned}$$

eq15.10

The contour of integration, L , is any upward-directed line parallel to the imaginary axis in the left half plane. We got the first equality by substituting in equation 16.1 and interchanging the s and t integrations. The second equality we got by noting

$$\begin{aligned}
 \int_0^t e^{-st' - i\omega t'} dt &= \int_0^t e^{-(s+i\omega)t'} dt' \\
 &= \frac{1}{s + i\omega} (1 - e^{-(s+i\omega)t}).
 \end{aligned}$$

If we allow $T_0(x) \neq 0$, then in evaluating the third integral of equation 16.1 we would also need to calculate the free space Green's function, as was done in chapter 10.

$$\begin{aligned}
 G(\mathbf{x}, t; \mathbf{x}', 0) &= \int \frac{ds}{2\pi i} e^{st} G(\mathbf{x}, \mathbf{x}'; \lambda = -s) \\
 &= \frac{e^{-(\mathbf{x}-\mathbf{x}')^2/4\kappa t}}{\sqrt{4\pi\kappa t}}. \quad (16.3)
 \end{aligned}$$

eq15.15

7 May p3

This applies to the special case of radiation in the infinite one-dimensional plane.

16.3 Evaluation of the Integrals

Recall that the Green's function can also be written as a bilinear expansion of the eigenfunctions. The general form of solution for equation 16.3 is

$$G(\mathbf{x}, t; \mathbf{x}', 0) = \begin{cases} \sum_n e^{-\lambda_n t} u_n(\mathbf{x}) u_n^*(\mathbf{x}') & \text{interior} \\ \int_0^\infty d\lambda' e^{-\lambda' t} \frac{1}{\pi} \text{Im} G(\mathbf{x}, \mathbf{x}', \lambda' + i\varepsilon) & \text{exterior.} \end{cases} \quad (16.4)$$

eq15.16
7 May p4

In the case when there is explicit time dependence, it may prove useful to integrate over t first, and then integrate over s . The expressions in equation 16.4 are particularly useful for large times. In this limit only a small range of λ_n must be used in the evaluation. In contrast, for short times, an expression like equation 16.3 is more useful.

If the functions T_{ext} , T_0 , q and ρ are spherically symmetric, then we only need the spherically symmetric part of the Green's function, G_0 . This was done in the second problem set. In contrast, for the problem presently being considered, the boundary conditions are arbitrary, but the sources are oscillating in time.

Ask Baker
about rotating.

Now we will simplify the integral expression in equation 16.2.

To evaluate equation 16.2, we will use the fact from chapter 10 that $G(x, x'; s)$ has the form

7 May p5

$$G(\mathbf{x}, \mathbf{x}', s) \propto \frac{e^{-\sqrt{s}}}{\sqrt{s}}.$$

Note that the second term in equation 16.2 is

$$\int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}'; \lambda = -s)}{s + i\omega} e^{-i\omega t} = 0,$$

because the integrand decays in the right-hand plane as $e^{-\sqrt{s}}$. Thus the fact that we have oscillating sources merely amounts to a change in denominator,

$$\int \frac{ds}{2\pi i} G(\mathbf{x}, \mathbf{x}'; \lambda = -s) e^{-st} \xrightarrow{\text{osc.}} \int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}'; \lambda = -s)}{s + i\omega} e^{-st}. \quad (16.5)$$

eq15osc

We thus need to evaluate the first term in equation 16.2. We close the contour in the left-hand s -plane, omitting the branch along the negative real axis, as shown in figure 16.1. By Cauchy's theorem, the closed contour gives zero:

fig15a

$$\int_L \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}'; \lambda = -s)}{s + i\omega} e^{-st} = 0.$$

The integrand vanishes exponentially along L_1 , L_5 . Over the small circle around the origin we have

$$\int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}'; \lambda = -s)}{s + i\omega} e^{-st} = e^{-i\omega t} G(\mathbf{x}, \mathbf{x}', \lambda = i\omega).$$

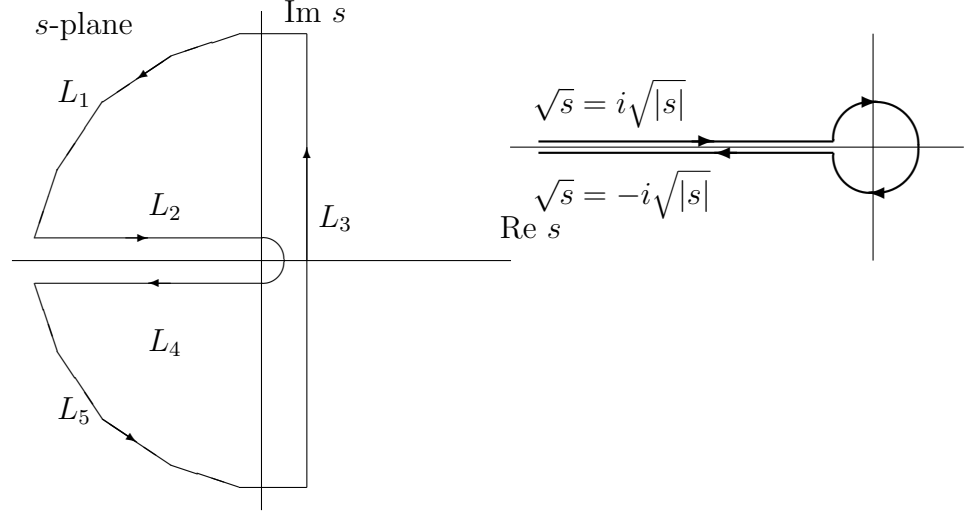


Figure 16.1: Closed contour around branch cut.

7 May p6

This looks like a steady state piece. We can use equation 16.4 to write

7 May p7

$$\int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}', \lambda = -s)}{s + i\omega} e^{st} = \begin{cases} \int = \sum_{n=0}^{\infty} e^{-\lambda_n t} \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{(\omega - \lambda_n)} & \text{discrete,} \\ \int_0^{\infty} d\lambda' \frac{e^{-\lambda' t} \frac{1}{\pi} \text{Im } G(\mathbf{x}, \mathbf{x}', \lambda' - i\varepsilon)}{i\omega - \lambda'} & \text{continuum.} \end{cases}$$

$$\begin{aligned} \int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}', \lambda = -s) e^{st}}{s + i\omega} &= \frac{1}{2\pi i} \left[\int_{-\infty}^0 \frac{ds'}{s' + i\omega} G(\mathbf{x}, \mathbf{x}'; \lambda = s' + i\varepsilon) \right. \\ &\quad \left. + \int_0^{\infty} \frac{ds'}{s' + i\omega} G(\mathbf{x}, \mathbf{x}'; \lambda = s' - i\varepsilon) \right]. \end{aligned}$$

7 May p8

Change variables

$$\lambda' = -s',$$

to obtain

$$\begin{aligned} \int \frac{ds}{2\pi i} \frac{G(\mathbf{x}, \mathbf{x}', \lambda = -s) e^{st}}{s + i\omega} &= \int_0^{\infty} \frac{d\lambda'}{i\omega - \lambda'} \frac{1}{2\pi i} [G(\mathbf{x}, \mathbf{x}', \lambda + i\varepsilon) - G(\mathbf{x}, \mathbf{x}', \lambda' - i\varepsilon)] \\ &= \int_0^{\infty} d\lambda' \frac{e^{-\lambda' t} \frac{1}{\pi} \text{Im } G(\mathbf{x}, \mathbf{x}', \lambda' - i\varepsilon)}{i\omega - \lambda'}. \end{aligned}$$

9 May p1

16.4 Physics of the Heat Problem

We have been looking at how to evaluate the general solution of the heat equation,

$$\begin{aligned} T(\mathbf{x}, t) = & \int_0^t dt' \int_R d\mathbf{x}' G(\mathbf{x}, t, \mathbf{x}', t') \rho(\mathbf{x}') \dot{q}(\mathbf{x}', t') \\ & + \int_0^t dt' \int_{\mathbf{x} \text{ in } R} ds' G(\mathbf{x}, t, \mathbf{x}', t') \alpha T_{\text{ext}}(s', t') \\ & + \int_{\mathbf{x} \text{ in } R} G(\mathbf{x}, t, \mathbf{x}', 0) \rho(\mathbf{x}') c_p(\mathbf{x}') T_0(\mathbf{x}'). \end{aligned}$$

We now look at the physics. We require the solution to satisfy the initial condition

$$T(\mathbf{x}, t) = T_0(\mathbf{x}) = 0 \quad \text{for } t = 0,$$

and the general regular boundary condition

$$[\kappa_T(\mathbf{x}) \hat{n} \cdot \nabla + \alpha] T(\mathbf{x}, t) = \alpha T_{\text{ext}}(\mathbf{x}, t) \quad \mathbf{x} \in S.$$

This boundary condition represents the balance between conduction and radiation.

16.4.1 The Parameter Θ

pr:Theta1

We can rewrite the regular boundary condition as

$$\begin{aligned} \hat{n} \cdot \nabla T|_{\mathbf{x} \text{ on } S} &= \frac{\alpha}{\kappa_{\text{Th}}} [T_{\text{ext}}(s, t) - T(\mathbf{x}, t)]_{\mathbf{x} \text{ on } S} \\ &= \Theta [T_{\text{ext}}(s, t) - T(\mathbf{x}, t)]_{\mathbf{x} \text{ on } S}, \end{aligned}$$

where $\Theta = \alpha/\kappa_{\text{Th}}$. The expression on the left hand side is the conduction in the body, while the expression on the right hand side is the radiation into the body. Thus, this equation is a statement of energy balance. The dynamic characteristic parameter in this equation is Θ , which has the dimensions of inverse distance:

$$\Theta = \frac{\alpha}{\kappa_{\text{Th}}} \sim \frac{1}{\text{distance}}.$$

We now consider large and small values of Θ .

9 May p2

Case 1: radiation important

In this region we have

$$\Theta = \frac{\alpha}{\kappa_{\text{th}}} \gg 1.$$

For this case we have radiation at large s and conduction is small, which means

$$T(\mathbf{x}, t) \approx T_{\text{ext}}(s, t) \quad \text{for } x \in S.$$

Case 2: heat flux occurs

This applies to the case

$$\Theta = \frac{\alpha}{\kappa_{\text{th}}} \ll 1.$$

Thus we take

$$\lim_{\substack{T_{\text{int}} \rightarrow \infty \\ \alpha \rightarrow 0}} \alpha T_{\text{int}} \equiv F(s, t),$$

where $F(s, t)$ is some particular heat flux at position s and time t . The boundary condition then becomes a fixed flux condition,

$$\kappa_{\text{th}} \hat{n} \cdot \nabla \mathbf{T}(\mathbf{x}, t)|_{\mathbf{x} \text{ on } s} = F(s, t).$$

9 May p3

HW comments
omitted

16.5 Example: Sphere

The region is the exterior region to a sphere with

$$T_{\text{ext}}(\theta, \varphi; t) = T_{\text{ext}}(t).$$

So we can write

$$G(\mathbf{x}, t, \mathbf{x}', t') = \frac{1}{4\pi} G_0(r, t, r', t').$$

So we just need to evaluate equation 16.1. We will get the typical functions of the theory of the heat equation.

We take the temperature T_{ext} on the surface of the sphere to be uniform in space and constant in time:

$$T_t(\theta, \varphi, t) = T_{\text{ext}}.$$

9 May p4

In this case plugging equation 16.3 into 16.1 yields

$$\begin{aligned} T(r, t) &= \int dt \int dx G(\mathbf{x}, t; \mathbf{x}', 0) \alpha T_{\text{ext}} \\ &= \int dt \int dx \frac{e^{-(\mathbf{x}-\mathbf{x}')^2/4\kappa t}}{\sqrt{4\pi\kappa t}} \alpha T_{\text{ext}} \\ &= T_{\text{ext}} \frac{a}{r} \operatorname{erfc} \left(\frac{r-a}{\sqrt{4\kappa t}} \right), \end{aligned}$$

where we define

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x dz e^{-z^2},$$

ask Baker

and

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty dz e^{-z^2}. \quad (16.6)$$

For short times, x is large, and for small times x is small and (Eq.14) is easy to evaluate. For large x we use integration by parts,

eq15.32

Mysterious equation omitted

$$\begin{aligned} \operatorname{erfc} x &= \frac{2}{\sqrt{\pi}} \int_x^\infty z dz e^{-z^2} \frac{1}{z} \\ &= \frac{2}{\sqrt{\pi}} \left[-\frac{1}{2} e^{-z^2} \frac{1}{z} - \int_x^\infty dz \left(-\frac{1}{2} e^{-z^2} \right) \left(-\frac{1}{z^2} \right) \right] \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty \frac{dz}{z^2} e^{-z^2} \right) \\ &= \frac{2e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{2x} - \frac{1}{4x^3} + \cdots \right). \end{aligned}$$

Thus we have a rapidly converging expansion for large x . For $x \ll 1$, we can directly place the Taylor series of e^{-z^2} inside the integral.

16.5.1 Long Times

We have standard diffusion phenomena. As $t \rightarrow \infty$, the solution goes $T(r, t) \xrightarrow{t \rightarrow \infty} \frac{a}{r} T_{\text{ext}}$. This is the steady state solution. It satisfies the conditions

$$\begin{aligned} \nabla^2 T(\mathbf{x}, t) &= 0, & |\mathbf{x}| > a, \\ T(\mathbf{x}, t) &= T_{\text{ext}}, & |\mathbf{x}| = a. \end{aligned}$$

If

$$x^2 = \frac{(r-a)^2}{4\kappa t} \gg 1$$

then we satisfy the steady state condition, and we can define τ by

$$\frac{(r-a)^2}{4\kappa\tau} = 1 \text{ and so } \tau = \frac{(r-a)^2}{4\kappa t}.$$

The variable τ is the characteristic time which determines the rate of diffusion. So for $t \gg \tau$, the temperature T is of the form of the steady state solution.

16.5.2 Interior Case

Having considered the region exterior to the sphere, we now consider the problem for the interior of the sphere. In particular, we take the surface source T_{ext} to have harmonic time dependence and arbitrary spatial independence:

$$T_{\text{ext}}(t, s) = e^{i\omega t} T_{\text{ext}}(s).$$

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We further assume that there are no volume source and that the internal temperature is initially zero:

$$\rho\dot{q}(\mathbf{x}, t) = 0, T(\mathbf{x}, t = 0) = 0.$$

In this case equation 16.2 reduces to

$$T(\mathbf{x}, t) = \int_{\mathbf{x} \text{ on } s} ds' \int_0^t dt' G(\mathbf{x}, t; \mathbf{x}', t') \alpha e^{i\omega t'} T_{\text{ext}}(t, s),$$

or

$$T(\mathbf{x}, t) = \int_{\mathbf{x} \text{ on } s} ds' \left[\alpha T_{\text{ext}}(s') \int_0^\infty dt' G(\mathbf{x}, t; \mathbf{x}', t') e^{-i\omega t'} \right].$$

This equation was computed previously for an external region. The solution was

$$\begin{aligned} T(\mathbf{x}, t) = & \int_{\mathbf{x}'} ds' \alpha T_{\text{ext}}(s') \left[e^{-i\omega t} G(\mathbf{x}, \mathbf{x}'; \lambda = i\omega) \right. \\ & \left. + \sum_m e^{-\lambda_n t} \frac{u_n(\mathbf{x}) u_n^*(\mathbf{x}')}{i\omega - \lambda_n} \right]. \end{aligned}$$

This holds for the discrete case, which occurs when the region is the interior of a sphere. For the continuous case, which is valid for the external problem, we have

$$T(\mathbf{x}, t) = \int_{\mathbf{x}'} ds' \alpha T_{\text{ext}}(s') \left[e^{-i\omega t} G(\mathbf{x}, \mathbf{x}'; \lambda = i\omega) + \frac{1}{\pi} \int_0^\infty d\lambda' e^{-\lambda' t} \frac{\text{Im } G(\mathbf{x}, \mathbf{x}'; \lambda = \lambda' + i\omega)}{i\omega - \lambda'} \right].$$

The first term in the bracketed expression is the steady state part of the response. The second term is the transient part of the response. These transient terms do not always vanish, as is the case in the fixed flux problem, in which there is a zero eigenvalue.

Many
HW comments
omitted

16.6 Summary

1. The general response to the time-dependent inhomogeneous heat equation is

$$T(\mathbf{x}, t) = \int_0^t dt' \int_R d\mathbf{x}' G(\mathbf{x}, t; \mathbf{x}', t') \rho(\mathbf{x}') \dot{q}(\mathbf{x}', t') + \int_0^t dt' \int_{\mathbf{x} \in S} ds' G(\mathbf{x}, t; \mathbf{x}', t') \alpha T_{\text{ext}}(s', t') + \int_R G(\mathbf{x}, t; \mathbf{x}', 0) \rho(\mathbf{x}') c_p(\mathbf{x}') T_0(\mathbf{x}').$$

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2. The boundary condition for the heat equation can be written

$$\hat{n} \cdot \nabla T|_{\mathbf{x} \text{ on } S} = \Theta [T_{\text{ext}}(s, t) - T(\mathbf{x}, t)]_{\mathbf{x} \text{ on } S},$$

where $\Theta = \alpha/\kappa_{\text{Th}}$. If $\Theta \gg 1$, then radiation is dominant, otherwise if $\Theta \ll 1$, then heat flux is dominant.

3. The temperature exterior to a fixed temperature circle is

$$T(r, t) = T_{\text{ext}} \frac{a}{r} \text{erfc} \left(\frac{r - a}{\sqrt{4\kappa t}} \right),$$

where

$$\text{erfc } x = 1 - \text{erf } x = \frac{2}{\sqrt{\pi}} \int_x^\infty dz e^{-z^2}.$$

16.7 References

See the references of chapter 10.

Chapter 17

The Wave Equation

Chapter Goals:

- State the free space Green's function in n -dimensions.
- Describe the connection between the even- and odd-dimensional Green's functions.

17.1 introduction

The Retarded Green's function for the wave equation satisfies

$$\left[-\tau \nabla^2 + \sigma \frac{\partial^2}{\partial t^2} \right] G(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$$

with the retarded boundary condition that $G_R = 0$ for $t < t'$. The solution to this equation is

$$G_R(\mathbf{x}, t; \mathbf{x}', t') = \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2), \quad (17.1)$$

where the integration path L is any line in the upper half plane parallel to the real axis and $R = |\mathbf{x} - \mathbf{x}'|$ and where $G(\mathbf{x}, \mathbf{x}'; \lambda)$ satisfies

$$[-\tau \nabla^2 - \sigma \lambda] G(\mathbf{x}, \mathbf{x}'; \lambda) = \delta(\mathbf{x} - \mathbf{x}'). \quad (17.2)$$

We denote the solution of equation 17.2 in n -dimensions as G_n . We

then have

$$G_1(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{ie^{i\sqrt{\lambda/c^2}R}}{2\tau\sqrt{\lambda/c^2}} \quad (17.3)$$

$$G_2(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{i}{4\tau} H_0^{(1)}(k, R) \quad (17.4)$$

$$G_3(\mathbf{x}, \mathbf{x}'; \lambda) = \frac{e^{i\sqrt{\lambda/c^2}R}}{4\pi\tau R}, \quad (17.5)$$

where $k = \sqrt{\lambda/c^2}$. It is readily verified that these three equations can be written in the more general form

$$G_n(R; \lambda) = \frac{i}{4\tau} \left(\frac{k}{2\pi R} \right)^{\frac{n}{2}-1} H_{\frac{n}{2}-1}^{(1)}(k, R).$$

The Fourier transform, equation 17.1, for the 3-dimensional case can be reduced to the Fourier transform for the one dimensional case, which we have already solved. The trick to do this is to rewrite the integral as a derivative with respect to the constant parameter R , and then pull the differential outside the integral.

$$\begin{aligned} G_R(\mathbf{x}, t, \mathbf{x}', t') &= \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_3(\mathbf{x}, \mathbf{x}'; \lambda = \omega^2) \\ &= \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{e^{(i\omega/c)R}}{4\pi\tau R} \\ &= \int_L \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \left(-\frac{1}{2\pi R} \right) \frac{\partial}{\partial R} \frac{i}{2\tau} e^{i\frac{\omega}{c}R} \\ &= -\frac{1}{2\pi R} \frac{\partial}{\partial R} \int_L \frac{d\omega}{2\pi} \frac{e^{i\frac{\omega}{c}R}}{2\tau} i e^{-i\omega(t-t')} \\ &= \frac{1}{2\pi R} \frac{\partial}{\partial R} G_1(\mathbf{x}, t; \mathbf{x}', t') \\ &= \frac{1}{2\pi R} \frac{\partial}{\partial R} \left[\frac{c}{2\tau} \theta(c(t-t') - R) \right] \end{aligned}$$

where the θ -function satisfies $d\theta(\mathbf{x}/d\mathbf{x} = \delta(\mathbf{x})$. Note that

$$f(a\mathbf{x}) = \frac{1}{|a|} \delta(\mathbf{x}).$$

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Thus we can write

$$\begin{aligned}
G_3(\mathbf{x}, t; \mathbf{x}', t') &= \frac{1}{2\pi R} \frac{\partial}{\partial R} \left[\frac{c}{2\tau} \theta(c(t - t') - R) \right] \\
&= \frac{1}{4\pi R\tau} \delta(c(t - t') - R/c) \\
&= \frac{1}{4\pi R\tau} \delta(t - t' - R/c).
\end{aligned}$$

Our result is then

$$G_3(R, t - t') = \frac{1}{\tau\pi R} \frac{\partial}{\partial R} G_1(R, t - t') = \frac{\delta(t - t' - R/c)}{4\pi R\tau}. \quad (17.6)$$

eq16.4

13 May p1

17.2 Dimensionality

17.2.1 Odd Dimensions

pr:oddDim1

Note that $H_{\frac{n}{2}-1}^{(1)}(k, R)$ is a trigonometric function for any odd integer. Thus for n odd, we get

13 May p2

$$G_n(R, t - t') = \left(-\frac{1}{2\pi R} \frac{\partial}{\partial R} \right)^{\frac{n-1}{2}} G_1(R, t - t')$$

and

$$G_n(R, \lambda) = \left(-\frac{1}{2\pi R} \frac{\partial}{\partial R} \right)^{\frac{n-1}{2}} G_1(R, \lambda).$$

Thus

$$G_n(R, \lambda) = \left(-\frac{i}{2\pi R} \right)^{\frac{n-1}{2}} G_1(R, \lambda)$$

for n odd. We also have

$$G_n(R, t - t') = \left(-\frac{1}{2\pi R} \frac{\partial}{\partial R} \right)^{\frac{n-3}{2}} G_3(R, t - t') \quad (17.7)$$

$$= \left(-\frac{1}{2\pi R} \frac{\partial}{\partial R} \right)^{\frac{n-3}{2}} \frac{\delta(t - t' - \frac{R}{c})}{4\pi\tau R}. \quad (17.8)$$

17.2.2 Even Dimensions

pr:evDim1

Recall that the steady state Green's function for 2-dimensions is

$$G_2(R, \lambda) = \frac{i}{4\pi} H_0^{(1)}(k, R).$$

If we insert this into equation 17.1 we obtain the retarded Green's function,

$$G_2(R, t - t') = \frac{c}{2\pi\tau} \frac{\theta(c(t - t') - R)}{\sqrt{c^2(t - t')^2 - R^2}}.$$

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17.3 Physics

There are two ways to define electrostatics. The first is by Gauss's law and the second is by its solution, Coulomb's law. The same relationship is true here.

17.3.1 Odd Dimensions

We consider the $n = 3$ -dimensional case,

$$G_3(R, t - t') = \frac{\delta(t - t' - R/c)}{4\pi R\tau}.$$

At time t the disturbance is zero everywhere except at the radius $R = c(t - t')$ from \mathbf{x}' . We only see a disturbance on the spherical shell.

17.3.2 Even Dimensions

In two dimensions the disturbance is felt at locations other than the surface of the expanding spherical shell. In two dimensions we have

13 May p3

$$G_2 = \frac{c}{2\tau} \frac{\theta[c(t - t') - R]}{\sqrt{c^2(t - t')^2 - R^2}} = \begin{cases} = 0 & R > c(t - t'), \\ \neq 0 & R < c(t - t'). \end{cases} \quad (17.9)$$

The case $G = 0$ for $R > c(t - t')$ makes sense since the disturbance has not yet had time to reach the observer. We also have

$$G_2 = \frac{c}{2\tau} \frac{\theta[c(t - t') - R]}{\sqrt{c^2(t - t')^2 - R^2}} \rightarrow \infty \quad \text{as } R \rightarrow c(t - t').$$

$$G_2$$

$$c(t - t') R$$

Figure 17.1: Radial part of the 2-dimensional Green's function.

Thus the maximum disturbance occurs at $R \rightarrow c(t - t')$. Finally, $G \neq 0$ for $R < c(t - t')$. Thus we have propagation at speed c , as well as all smaller velocities. This is called a wake. The disturbance is shown in figure 17.1. We have not yet given a motivation for why $G_2 \neq 0$ for $R > c(t - t')$. This will be done in the next section, where we will also give an alternative derivation of this result. *fig16a*

17.3.3 Connection between GF's in 2 & 3-dim

We now calculate the Green's function in 2-dimensions using the Green's function in 3-dimensions. This will help us to understand the difference between even and odd dimensions. Consider the general inhomogeneous wave equation in three dimensions,

$$\left[-\tau \nabla_3^2 + \sigma \frac{\partial^2}{\partial t^2} \right] u(\mathbf{x}, t) = \sigma f(\mathbf{x}, t). \quad (17.10)$$

From our general theory we know that the solution of this equation can be written in terms of the Green's function as *eq16.1*

$$u(\mathbf{x}, t) = \int_0^t dt' \int d\mathbf{x}' G_3(\mathbf{x}, t; \mathbf{x}', t') \sigma f(\mathbf{x}', t'). \quad (17.11)$$

We now consider a particular source,

eq16.2

$$\sigma f(\mathbf{x}', t') = \delta(x') \delta(y') \delta(t - t_0).$$

This corresponds to a line source along the z -axis acting at time $t = t_0$.

What equation does u satisfy for this case? The solution will be completely independent of z : $u(\mathbf{x}, t) = u(x, y, t) = u(x^2 + y^2, t) = u(\rho, t)$ 13 May p6

where $\rho = x^2 + y^2$, and the second equality follows from rotational invariance. For this case equation 17.10 becomes

$$\left[-\nabla_2^2 + \frac{\sigma}{\tau} \frac{\partial}{\partial t^2} \right] u(x, y, t) = \frac{1}{\tau} \delta(x) \delta(y) \delta(t - t_0).$$

Thus

$$u(x, y, t) = G_2(\rho, t - t_0),$$

where G_2 was given in equation 17.9. We should be able to get the same result by plugging the expression for G_3 , equation 17.6, in equation 17.11. Thus we have

$$u(\mathbf{x}, t) = \int_0^t dt' \int dx' dy' dz' \frac{1}{4\pi R} \delta(t - t' - R/c) \delta(x') \delta(y') \delta(t' - t_0) \quad (17.12)$$

eq16.6 Now let $u(\mathbf{x}, t) = u(x, y, 0, t) = G_2(\rho, t - t')$ on the left hand side of
13 May p7 equation 17.12 and partially evaluate the right hand side to get

$$G_2(\rho, t - t_0) = \int_0^t dt' \int_{-\infty}^{\infty} dz' \frac{1}{4\pi\tau} \frac{\delta(t - t_0 - R/c)}{\sqrt{\rho^2 + z'^2}}. \quad (17.13)$$

eq16.7 The disturbance at time t at the field point will be due to contributions
at $z = 0$ from $\rho = c(t - t')$. We also have disturbances at farther
distances which were emanated at an earlier time. This is shown in
fig16b figure 17.2.

16 May p1 Note that only the terms at z' contribute, where $z'^2 + \rho^2 = c^2(t - t_0)^2$.
So we define

$$z' = z_{\pm} = \pm \sqrt{c^2(t - t_0)^2 - \rho^2}$$

We now consider the value of G_2 using equation 17.13 for three different regions.

- $G_2 = 0$ if $\rho > c(t - t_0)$ for a signal emitted at z . This is true since a signal emitted at any z will not have time to arrive at ρ since it travels at velocity c .
- If $\rho = c(t - t_0)$, then the signal emitted from the point $z = 0$ at time t_0 arrives at ρ at time t . Thus $z_{\pm} = 0$.
- Finally if $\rho < c(t - t_0)$, then the signals emitted at time $t = t_0$ from the points $z = z_{\pm}$ arrive at time t .

This is the origin of the wake.

17.5 Summary

1. The free space Green's function in n -dimensions is

$$G_n(R; \lambda) = \frac{i}{4\tau} \left(\frac{k}{2\pi R} \right)^{\frac{n}{2}-1} H_{\frac{n}{2}-1}^{(1)}(k, R).$$

2. The connection between the fact that the 3-dimensional Green's function response propagates on the surface of a sphere and the fact the the 2-dimensional Green's function response propagates inside of a cylinder is illustrated.

17.6 References

See [Fetter80] and [Stakgold67]. This chapter is mostly just an exploration of how the number of dimensions affects the solution form.

Chapter 18

The Method of Steepest Descent

pr:StDesc1

Chapter Goals:

- Find the solution to the integral $I(\omega) = \int_C dz e^{\omega f(z)} g(z)$.
- Find the asymptotic form of the Gamma function.
- Find the asymptotic behavior of the Hankel function.

Suppose that we integrate over a contour C such as that shown in figure 18.1:

$$I(\omega) = \int_C dz e^{\omega f(z)} g(z). \quad (18.1)$$

We want to find an expression for $I(\omega)$ for large ω . Without loss of generality, we take ω to be real and positive. This simply reflects the choice of what we call $f(z)$. The first step will be to take the indefinite integral. The second step will then be to deform the contour C into a contour C_0 such that

$$\left. \frac{df}{dz} \right|_{z=z_0} = 0$$

where z_0 lies on the contour C_0 .

In order to perform these operations we will first digress to a review

Figure 18.1: Contour C & deformation C_0 with point z_0 .

fig17.1

of the methods of complex analysis which are needed to compute this integral. Then we shall explicitly solve the integral.

18.1 Review of Complex Variables

pr:anal2

Let $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$ where $f(z)$ is analytic on the region which we are considering. In general a function f of the complex variable z is *analytic* (or *holomorphic*) at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighborhood of z_0 , and a function f is said to be analytic in a region R if it is analytic at each point in R . In this case we have:

$$\frac{df}{dz} = \frac{d}{dz}(u + iv) = \frac{du}{dz} + i\frac{dv}{dz}.$$

Since the function is analytic, its derivative is independent of the path of approach.

If we differentiate with respect to an infinitesimal change $dz = dx$, we get

$$\frac{df}{dz} = \frac{du}{dx} + i\frac{dv}{dx}, \quad (18.2)$$

eq17.1

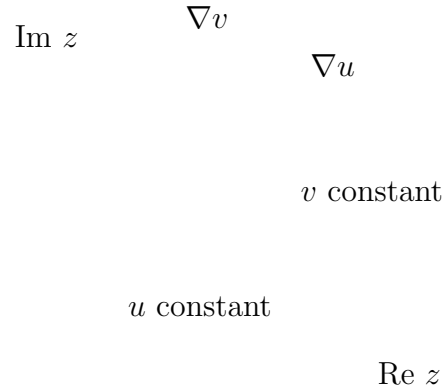
and if we differentiate with respect to an infinitesimal change $dz = idy$, we get

$$\frac{df}{dz} = \frac{du}{d(iy)} + i\frac{dv}{d(iy)} = -i\frac{du}{dy} + \frac{dv}{dy}. \quad (18.3)$$

eq17.2

pr:CREq1

By comparing equations 18.2 and 18.3 and separating the resulting equation into real and imaginary parts we get the Cauchy-Riemann equations:

Figure 18.2: Gradients of u and v .

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{dv}{dx} = -\frac{du}{dy}.$$

These facts allow us to make the following four observations about differentiation on the complex plane:

Observation 1. The gradient of a complex valued function is depicted in figure 18.2 for an integral curve of an analytic function. *fig17.2*

The product of gradients is given by the equation

$$\nabla u \cdot \nabla v = \frac{du}{dx} \frac{dv}{dx} + \frac{du}{dy} \frac{dv}{dy} = 0.$$

The last equality follows from the Cauchy-Riemann equations. This means that the lines for which u is constant are perpendicular (i.e., orthogonal) to the lines for which v is constant.

Observation 2. For the second derivatives we have the following relations:

$$\frac{d^2u}{dx^2} = \frac{d}{dx} \left(\frac{du}{dx} \right) = \frac{d}{dx} \frac{dv}{dy} \quad (18.4)$$

and

$$\frac{d^2u}{dy^2} = \frac{d}{dy} \left(\frac{du}{dy} \right) = \frac{d}{dy} \left(-\frac{dv}{dx} \right). \quad (18.5) \quad \text{eq17.3}$$

The differentials commute, so by combining equations 18.4 and 18.5 *eq17.4* we get

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

$$u = \operatorname{Re} f(z)$$

$$x = \operatorname{Re} z$$

$$y = \operatorname{Im} z$$

Figure 18.3: $f(z)$ near a saddle-point.

and similarly

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0.$$

pr:LapEq1

This means that analytic functions satisfy Laplace's equation.

Observation 3. From Observation 2 we find that:

$$\text{If } \frac{d^2u}{dx^2} > 0 \text{ then } \frac{d^2u}{dy^2} < 0. \quad (18.6)$$

eq17.5

Thus we cannot have a maximum or a minimum of both u and v occur anywhere in the complex plane. The point $z_0 = x_0 + iy_0$ for which

$$\frac{du}{dx}\bigg|_{z_0} = 0 \quad \text{and} \quad \frac{du}{dy}\bigg|_{z_0} = 0$$

is called a saddle point. The Cauchy–Riemann equations and equation 18.6 imply that if $df/dz = 0$ at z_0 , then z_0 is a saddle point of both $u(x, y)$ and $v(x, y)$. This is illustrated in figure 18.3.

fig17.3

Observation 4. For an analytic function $f = u + iv$ and a differential $d\mathbf{l}$ we have

$$\begin{aligned} df &= d\mathbf{l} \cdot \nabla f \\ &= d\mathbf{l} \cdot \nabla u + i d\mathbf{l} \cdot \nabla v. \end{aligned}$$

Note that $|df/dz|$ is independent of the direction of $d\mathbf{l}$ due to analyticity. Suppose that we chose $d\mathbf{l}$ to be perpendicular to ∇u . In this case $d\mathbf{l} \cdot \nabla v = 0$, so

$$df = d\mathbf{l} \cdot \nabla u \quad \text{for } d\mathbf{l} \perp \nabla u.$$

As the magnitude of f changes, the change $d\mathbf{l} \cdot \nabla u$ is purely real, since $u(x, y)$ is real. Thus the real part of f has maximum change in the direction where $d\mathbf{l} \cdot \nabla v = 0$, since $|df/dz|$ is independent of direction. Therefore $d\mathbf{l} \cdot \nabla v = 0$ gives the path of either steepest descent or steepest ascent. The information given so far is insufficient to determine which.

18.2 Specification of Steepest Descent

We want to evaluate the integral from equation 18.1,

$$I(\omega) = \int_C dz e^{\omega f(z)} g(z),$$

for ω large. We take ω to be real and positive. In the previous section we wrote $f(z) = u(z) + iv(z)$. Thus we want to know $\text{Re}(f(z))$ in order to determine the leading order behavior of $I(\omega)$ for $\omega \gg 1$.

To solve for $I(\omega)$ we deform $C \rightarrow C_0$ such that most of the contribution of the integral when $\omega \gg 1$ comes from a small region on C_0 . Thus we need to make an optimal choice of contour. We want $df/dz = 0$ at some point $z = z_0$ on the deformed contour C_0 . We parameterize C_0 with the line

$$z(\tau) = x(\tau) + iy(\tau).$$

We want the region of the curve where $u(\tau) = \text{Re}(f(\tau))$ to be as localized as possible. Thus we want the contour to run in the direction where $u(\tau)$ has maximal change. As we saw at the end of the previous section, this occurs when $v(z(\tau)) = v(\tau)$ remains constant. So our deformed contour C_0 has the property that

$$v(\tau) = \text{a constant on } C_0. \quad (18.7)$$

This will uniquely determine the contour.

eq17.6

Note that we assume there is only one point where $df/dz = 0$. If there were more than one such point, then we would merely repeat this process at the new point and add its contribution.

Equation 18.7 is equivalent to the condition

$$\text{Im}[f(z(\tau)) - f(z_0)] = 0.$$

The path for which this condition is satisfied is also the one for which

$$\operatorname{Re}[f(z) - f(z_0)] = u(z) - u(z_0)$$

changes most rapidly.

We want to evaluate $I(\omega)$ for ω large. Recalling the condition for a local maximum or minimum that $df/dz = 0$, we note that it is useful to rewrite the integral defined in equation 18.1 as

$$I(\omega) = \int_C dz e^{\omega f(z)} g(z) = e^{\omega f(z_0)} \int_{C_0} dz e^{\omega(f(z) - f(z_0))} g(z).$$

Note that since $f(z)$ is an analytic function, the integral over C_0 is equal to the integral over C . The main contribution is at the maximum of the difference $f(z) - f(z_0)$. We want to find the curve with the maximum change, which has a local maximum at z_0 , which means we want the quantity $f(z) - f(z_0)$ to be negative. Thus we want the curve along which

$$\operatorname{Re}[f(z) - f(z_0)] = u(z) - u(z_0)$$

changes most rapidly and is negative. This is called the curve of steepest descent. This condition specifies which of the two curves specified by $\operatorname{Im}[f(z(\tau)) - f(z_0)] = 0$ we choose: we choose the path of steepest descent.

18.3 Inverting a Series

pr:invSer1

We choose the parameterization

$$f(z) - f(z_0) \equiv -\tau^2$$

so we get

$$z(\tau = 0) = z_0.$$

Note that τ is real since $\Delta v(\tau) = 0$ along the curve and $f(z) < f(u_0)$.

We need to invert the integral. Expand $f(z) - f(z_0)$ in a power series about z_0 :

$$f(z) - f(z_0) = \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \dots = -\tau^2. \quad (18.8)$$

eq17.7

We also get

$$z - z_0 = \sum_{n=1}^{\infty} a_n \tau^n = a_1 \tau + a_2 \tau^2 + a_1 \tau^3 + \dots \quad (18.9)$$

Note that there is no constant term in this series. This is because $\tau = 0$ eq17.8 implies $z - z_0 = 0$. Thus, if we had an $n = 0$ term, we couldn't satisfy this stipulation.

Plug equation 18.9 into 18.8:

$$-\tau^2 = \frac{f''(z_0)}{2!} \left(\sum_{n=1}^{\infty} a_n \tau^n \right)^2 + \frac{f'''(z_0)}{3!} \left(\sum_{n=1}^{\infty} a_n \tau^n \right)^3.$$

To calculate a_1 , forget the terms $(f'''(z_0)/3!)(z - z_0)^3$ on. The calculation of a_2 includes this term and the calculation of a_3 includes the following term. Thus

$$-\tau^2 = \frac{f''(z_0)}{2!} a_1^2 \tau^2 + \mathcal{O}(\tau^3). \quad (18.10)$$

Now let

$$\frac{f''(z_0)}{2!} \equiv R e^{+i\theta}.$$

eq17.9

Plugging this into equation 18.10 and canceling τ^2 yields

$$-1 = a_1^2 R e^{i\theta}$$

so

$$a_1^2 = \frac{e^{i\pi - i\theta}}{R}$$

where $-1 = e^{i\pi}$. So

$$a_1 = \frac{1}{\sqrt{R}} e^{i(-\frac{\theta}{2} \pm \frac{\pi}{2})}. \quad (18.11)$$

The calculation of the a_i 's is the only messy part involved in finding subsequent terms of the inverted series. For our purposes, it is sufficient to have calculated a_1 . The \pm in equation 18.11 gives us two curves for the first term: eq17.10

$$z - z_0 \approx a_1 \tau = \frac{\tau}{\sqrt{R}} e^{i(-\theta \pm \pi)/2}.$$

We now assume that we have calculated the whole series, and use the series to rewrite $I(\omega)$. We now take our integral

$$I(\omega) = e^{\omega f(z_0)} \int_{C_0} dz e^{\omega(f(z)-f(z_0))} g(z),$$

and make a variable substitution

$$dz = \frac{dz}{d\tau} d\tau$$

to obtain

$$I(\omega) = e^{\omega f(z_0)} \int_{\tau_-}^{\tau_+} d\tau e^{-\omega \tau^2} \frac{dz}{d\tau} g(z(\tau)),$$

where τ_+ and τ_- are on the curve C_0 on opposite sides of $\tau = 0$. We expand the $z(\tau)$ in the function $g(z(\tau))$ as

$$z = a_1 \tau + a_2 \tau^2 + \dots$$

and thus

$$\frac{dz}{d\tau} g(z(\tau)) = \sum_{n=0}^{\infty} c_n \tau^n, \quad (18.12)$$

eq17.11 where the c_n can be determined from the a_n and $g(z(\tau))$. Thus we can write

$$I(\omega) = e^{\omega f(z_0)} \sum_n \int_{\tau_-}^{\tau_+} d\tau e^{-\omega \tau^2} c_n \tau^n.$$

Thus, with no approximations being made so far, we can assert

$$I(\omega) = e^{\omega f(z_0)} \sum_n c_n \int_{\tau_-}^{\tau_+} d\tau e^{-\omega \tau^2} \tau^n.$$

Now let $\tau_- \rightarrow \infty$ and $\tau_+ \rightarrow \infty$. Our integral becomes

$$I(w) = e^{w f(z_0)} \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} d\tau e^{-w \tau^2} \tau^n.$$

This is an elementary integral. We know

$$\int_{-\infty}^{\infty} d\tau e^{-w \tau^2} = \sqrt{\frac{\pi}{w}},$$

$$\int_{-\infty}^{\infty} d\tau \tau^2 e^{-w\tau^2} = \frac{d}{d\omega} \int_{-\infty}^{\infty} d\tau e^{-w\tau^2} = \frac{d}{d\omega} \sqrt{\frac{\pi}{w}} = \frac{\sqrt{\pi}}{2\omega^{3/2}},$$

(this is called differentiating with respect to a parameter), and similarly

$$\int_{-\infty}^{\infty} d\tau \tau^{2m} e^{-w\tau^2} = \left(\frac{d}{d\omega} \right)^n \int_{-\infty}^{\infty} d\tau e^{-w\tau^2} = \sqrt{\pi} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m \omega^{(2m+1)/2}}.$$

Since odd n gives zero by symmetry, we have

$$I(w) = e^{wf(z_0)} \sum_{n=0,2,4,\dots}^{\infty} c_n \int_{-\infty}^{\infty} d\tau e^{-w\tau^2} \tau^n.$$

All this gives

$$I(w) = e^{wf(z_0)} \left[c_0 \sqrt{\frac{\pi}{w}} + \frac{c_2}{2} \frac{\sqrt{\pi}}{w^{3/2}} + \sqrt{\pi} \sum_{m=2}^{\infty} c_{2m} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m w^{(2m+1)/2}} \right].$$

The term $c_0 \sqrt{\pi/\omega}$ corresponds to Sterling's formula and the term $\frac{c_2}{2} \frac{\sqrt{\pi}}{w^{3/2}}$ is the first correction to Sterling's formula. The only computation remaining is the $dz/d\tau$ in equation 18.12.

18.4 Example 1: Expansion of Γ -function

pr:Gamma1

We want to evaluate the integral

$$I(w) = \int_0^{\infty} e^{-t} t^w dt.$$

18.4.1 Transforming the Integral

We want to get this equation into the standard form. We make an elementary transformation to get it into the form

$$\int dz e^{wf(z)} g(z).$$

We substitute $t = zw$ to get

$$I(w) = \int_0^{\infty} dz e^{-wz} (zw)^w \quad (18.13)$$

$$= w^{w+1} \int_0^{\infty} dz e^{w[\log z - z]} \quad (18.14)$$

For $g = 1$ we have

$$f(z) = \log z - z$$

and we have

$$\frac{df}{dz} = \frac{1}{z} - 1 = 0 \quad \text{at } z = 1.$$

This is a saddle point. We chose to define φ on the interval

$$-\pi < \varphi < \pi$$

so that

$$z = re^{i\varphi}, \quad \log z = \log r + i\varphi.$$

This is analytic everywhere except the negative real axis, which we don't need.

18.4.2 The Curve of Steepest Descent

Since we know the saddle point, we can write

$$f(z) - f(z_0) = \log z - z + 1.$$

So we just need to calculate

$$0 = \operatorname{Im}[f(z) - f(z_0)] \tag{18.15}$$

$$= \varphi - r \sin \varphi. \tag{18.16}$$

We expect the lines of steepest ascent and descent passing through z_0 to be perpendicular to each other. The two solutions of this equation correspond to these curves. The solution $\varphi = 0$ gives a line on the positive real axis. The other solution is

$$r = \frac{\varphi}{\sin \varphi} \tag{18.17}$$

$$\approx 1 + \frac{\varphi^2}{6} \quad \text{for } \varphi \ll 1. \tag{18.18}$$

We haven't yet formally shown which one is the line of steepest ascent and descent. This is determined by looking at the behavior of $f(z)$ on each line.

By looking at $\log z - z + 1$ we see that $f(z) - f(z_0)$ can be written

$$\begin{aligned} f(z) - f(z_0) &= \log z - (z - 1) \\ &= -\tau^2 \\ &= \sum_n c_n (z - 1)^n. \end{aligned} \quad (18.19)$$

We have to invert this in order to get the asymptotic expansion of the gamma function. We expand equation 18.19 in a power series, after noting that $\log z = \log[(z - 1) + 1]$:

$$-\tau^2 = -\frac{1}{2}(z - 1)^2 + \frac{1}{3}(z - 1)^3 - \frac{1}{4}(z - 1)^4 + \dots$$

We plug in

$$z(\tau) - 1 = A\tau + B\tau^2 + C\tau^3 + \mathcal{O}(\tau^4)$$

$$\begin{aligned} -\tau^2 &= -\frac{1}{2}(A\tau + B\tau^2 + C\tau^3)^2 + \frac{1}{3}(A\tau + B\tau^2)^3 - \frac{1}{4}(A\tau)^4 + \mathcal{O}(\tau^5) \\ &= -\frac{1}{2}A^2\tau^2 - A\left(B - \frac{A^2}{3}\right)\tau^3 - \left(\frac{B^2}{2} + AC - A^2B + \frac{A^4}{4}\right)\tau^4 + \mathcal{O}(\tau^5). \end{aligned}$$

Comparing coefficients on the left and right hand side, we get

$$\begin{aligned} \tau^2 : \quad & A^2 = 2 \\ \tau^3 : \quad & A(B - A^2/3) = 0 \\ \tau^4 : \quad & \frac{B^2}{2} + AC - A^2B + \frac{A^4}{4} = 0. \end{aligned}$$

This method is called inverting the power series. We find

$$A = \sqrt{2},$$

$$C = \sqrt{2}/8.$$

The positive roots were chosen for convenience. Now we calculate what the c_n 's are from

$$\frac{dz}{d\tau}g(z(\tau)) = \sum_{n=0}^{\infty} c_n \tau^n.$$

Since $g(z) = 1$ and

$$\frac{dz}{d\tau} = A + 2B\tau + 3C\tau^2 + \dots,$$

we know that

$$C_0 = \sqrt{2},$$

$$C_2 = \frac{\sqrt{2}}{6}.$$

Finally, we plug in these values:

$$\begin{aligned} \int_0^\infty e^{-t} t^w dt &= I(w) \\ &= e^{wf(z_0)} \left[c_0 \sqrt{\frac{\pi}{w}} + \frac{c_2}{2} \frac{\sqrt{\pi}}{w^{3/2}} + \dots \right] \\ &= e^{-w} \left[\sqrt{\frac{2\pi}{w}} + \frac{\sqrt{2\pi}}{12w^{3/2}} + \dots \right], \end{aligned}$$

which agrees with Abramowitz & Stegun, formula 6.1.37.

18.5 Example 2: Asymptotic Hankel Function

pr:Hankel1

We want to find the asymptotic form of the Hankel function, starting with the integral representation

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty + \pi i} e^{z \sinh w - \nu w} dw$$

fig17.4

The contour of integration is the figure 18.4. The high index and argument behavior of the Hankel function

$$H_\nu^{(1)}(z)$$

are important in high energy scattering. The index ν is related to the effect of an angular momentum barrier, and z to an energy barrier. In this equation ν is an arbitrary complex number and z is an arbitrary complex number in a certain strip of the plane.



Figure 18.4: Defining Contour for the Hankel function.

We now relabel the Hankel function as $H_p^{(1)}(z)$, where

$$\frac{p}{z} = \cos \omega_0.$$

Note that p/z is real and $0 < p/z < 1$, which implies $0 < \omega_0 < \pi/2$. So

$$H_p(z) = \frac{1}{\pi i} \int_C dz e^{zf(w)} g(w)$$

where $g(w) = 1$, with

$$f(w) = \sinh w - pw/z = \sinh w - w \cos \omega_0.$$

To examine asymptotic values $|z| \gg 1$ with ω_0 fixed, we want to deform the contour so that it goes through a saddle point. Using the usual method, we have

$$\frac{df(w)}{dw} = \cosh w - \cos \omega_0 = 0.$$

We define

$$w_0 = i\omega_0$$

so that

$$\cosh w_0 \cosh i\omega_0 = \cos \omega_0.$$

Thus

$$f(w = i\omega) = \sinh i\omega_0 - i\omega_0 \cos \omega_0 = i[\sin \omega_0 - \omega_0 \cos \omega_0]$$

so that

$$f(w) - f(w_0) = \sinh w - w \cos \omega_0 - i[\sin \omega_0 - \omega_0 \cos \omega_0].$$

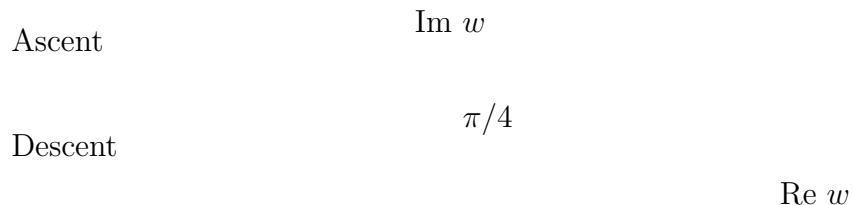


Figure 18.5: Deformed contour for the Hankel function.

We want to find out what the curves are. Note that

$$\left. \frac{d^2 f(w)}{dw^2} \right|_{w=i\omega_0} = \sinh w \Big|_{w=i\omega_0} = i \sin \omega_0$$

so we can write

$$-\tau^2 = f(w) - f(i\omega_0) = \frac{1}{2}i(\sin \omega_0)(w - i\omega_0)^2 + \dots$$

To invert this series, we write

$$w - i\omega_0 = A\tau + B\tau^2 + C\tau^3 + \dots$$

For now, we are just interested in the leading order term. So

$$-\tau^2 = \frac{1}{2}A^2\tau^2 i(\sin \omega_0)$$

which implies

$$A^2 = \frac{2i}{\sin \omega_0}.$$

Recall that we are looking for the tangent of C_0 at w_0 . Thus we have

$$A = \pm e^{i\pi/4} \sqrt{\frac{2}{\sin \omega_0}},$$

so

$$w - i\omega_0 = \pm \sqrt{\frac{2}{\sin \omega_0}} e^{i\pi/4} \tau.$$

fig17.5

The deformed curve C_0 has the form shown in figure 18.5. The picture neglects to take into account higher order terms. We choose the plus

sign to get the direction correct. Note that the curve of steepest ascent is obtained by a rotation of $\pi/2$ of the tangent, not the whole curve.

In the power series

$$g(z(\tau)) \frac{dz}{d\tau} = \sum_{n=0}^{\infty} c_n \tau^n$$

where $g(z(\tau)) = 1$, we have

$$\frac{dw}{d\tau} = A + \text{higher order terms},$$

so

$$c_0 = A = e^{i\pi/4} \sqrt{\frac{2}{\sin \omega_0}}.$$

Note that

$$\sin \omega_0 = \sqrt{1 - \frac{p^2}{z^2}} = \frac{1}{z} \sqrt{z^2 - p^2}.$$

Usually we consider $z \gg p$, so that

$$\frac{1}{z} \sqrt{z^2 - p^2} = 1.$$

The equation for C_0 comes from

$$\text{Im} [f(w) - f(w_0)] = 0$$

where $w = u + iv$. So in the equation

$$\text{Im} [f(w) - f(w_0)] = \cosh u \sin v - v \cos \omega_0 - (\sin \omega_0 - \omega_0 \cos \omega_0) = 0.$$

Thus,

$$u \rightarrow +\infty \quad \text{implies} \quad \cosh u \rightarrow +\infty \quad \text{so} \quad v = 0, \pi,$$

$$u \rightarrow -\infty \quad \text{implies} \quad \cosh u \rightarrow -\infty \quad \text{so} \quad v = 0, \pi,$$

This gives the line of steepest ascent and descent. The orientation of the curves of ascent and descent are shown in figure 18.6

fig17.6

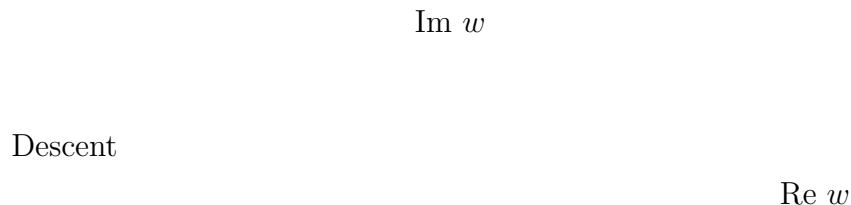


Figure 18.6: Hankel function contours.

18.6 Summary

1. The asymptotic solution of the integral

$$I(\omega) = \int_C dz e^{\omega f(z)} g(z)$$

is

$$I(w) = e^{\omega f(z_0)} \left[c_0 \sqrt{\frac{\pi}{w}} + \frac{c_2}{2} \frac{\sqrt{\pi}}{w^{3/2}} + \sqrt{\pi} \sum_{m=2}^{\infty} c_{2m} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m w^{(2m+1)/2}} \right].$$

2. The asymptotic expansion for the Gamma function is

$$\int_0^{\infty} e^{-t} t^w dt = e^{-w} \left[\sqrt{\frac{2\pi}{w}} + \frac{\sqrt{2\pi}}{12w^{3/2}} + \cdots \right].$$

3. The asymptotic behavior of the Hankel function is discussed in section 17.4.

18.7 References

See [Dennery], as well as [Arfken85].

Chapter 19

High Energy Scattering

Chapter Goals:

- Derive the fundamental integral equation of scattering.
- Derive the Born approximation.
- Derive the integral equation for the transition operator.

25 May p1

The study of scattering involves the same equation (Schrödinger's) as before, but subject to specific boundary conditions. We want solutions for the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t) = H \Psi(\mathbf{x}, t), \quad (19.1)$$

where the Hamiltonian is

eq18.1

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) = H_0 + V.$$

We look for steady state solutions of the form

$$\Psi(\mathbf{x}, t) = e^{-i(E/\hbar)t} \Psi_E(\mathbf{x}), \quad (19.2)$$

using the association $E = \hbar\omega$. In particular we want $E > 0$ solutions, *eq18.3* since the solutions for $E < 0$ are bound states. By substituting equation *pr:bound1*

19.2 into 19.1 we find that Ψ_E satisfies

$$(E - H)\Psi_E(\mathbf{x}) = 0. \quad (19.3)$$

eq18.4 The boundary condition of scattering requires that the wave function be of the form

$$\Psi_E(\mathbf{x}) = e^{i\mathbf{k}_i \cdot \mathbf{x}} + \Psi_S(\mathbf{x}) \quad (19.4)$$

eq18.5 where the incident wave number \mathbf{k}_i is

$$\mathbf{k}_i = \sqrt{\frac{2mE}{\hbar^2}} \hat{e}_z.$$

We interpret equation 19.4 as meaning that the total wave function is the sum of an incident plane wave $e^{i\mathbf{k}_i \cdot \mathbf{x}}$ with wavelength $\lambda = 2\pi/|\mathbf{k}_i|$, and a wave function due to scattering. This solution is illustrated by the following picture:

$$\psi_E(\mathbf{x}) = \underbrace{\begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array}}_{e^{i\mathbf{k} \cdot \mathbf{x}}} + \underbrace{\begin{array}{c} \text{(())} \\ \text{(())} \\ \bullet \\ \text{(())} \\ \text{(())} \end{array}}_{\psi_S(\mathbf{x})}$$

25 May p2

At distances far from the scatterer ($r \gg 1$), the scattered wave function becomes¹

$$\Psi_s(\mathbf{x}) = \frac{e^{ikr}}{r} f(\mathbf{k}_i, \mathbf{k}_f; E) \quad \text{for } r \gg 1 \quad (19.5)$$

eq18.7 where the final wave number \mathbf{k}_f is

$$\mathbf{k}_f = \frac{\mathbf{p}_f}{\hbar} = \hat{\mathbf{x}} \sqrt{\frac{2mE}{\hbar^2}}.$$

The unit vector \hat{x} simply indicates some arbitrary direction of interest. Equation 19.5 is the correct equation for the scattered wave function. The angular function $f(\mathbf{k}_i, \mathbf{k}_f; E)$ is called the form factor and contains the physical information of the interaction.

¹Again, see most any quantum mechanics text.

19.1. FUNDAMENTAL INTEGRAL EQUATION OF SCATTERING 283

For the case of spherical symmetry

$$f(\mathbf{k}_f, \mathbf{k}_i; E) = f(\mathbf{k}_f \cdot \mathbf{k}_i; E).$$

where $\mathbf{k}_f \cdot \mathbf{k}_i = k_f k_i \cos \theta$.

We now would like to formulate the scattering problem for an arbitrary interaction. Thus we look at the relation of the above formulation to Green's functions. The form of equation 19.3 appropriate for Green's functions is

$$(E - H)G(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}').$$

Note the minus sign (used by convention) on the left hand side of this equation. We solved equation 19.3 by writing (for the asymptotic limit $|\mathbf{x}'_s| \rightarrow \infty$)

$$G \rightarrow -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \Psi_E(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} [e^{i\mathbf{k} \cdot \mathbf{x}} + \Psi_s(\mathbf{x})]$$

where $\Psi_E(\mathbf{x})$ satisfies

$$(E - H)\Psi_E(\mathbf{x}) = 0.$$

The Green's function holds asymptotically since $\delta(\mathbf{x} - \mathbf{x}') \rightarrow 0$ as $|\mathbf{x}'| \rightarrow \infty$. This is the solution of the Schrödinger equation which has the boundary condition of scattering.

This needs
fixin'
pr:bcos1

19.1 Fundamental Integral Equation of Scattering

The equation for a general Green's function is

G 5/25/88
j25 May p3

$$(E - H)G(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}'). \quad (19.6)$$

Since $H = H_0 + V$, where $H_0 = -\hbar^2 \nabla^2 / 2m$, the free space Green's function satisfies

eq18.13

$$(E - H_0)G_0(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}').$$

As we have seen, the solution to this equation is

See also Jack-
son, p.224.

$$G_0 = -\frac{m}{2\pi\hbar^2} \frac{e^{ikR}}{R}, \quad (19.7)$$

where $R = |\mathbf{x} - \mathbf{x}'|$. We now convert equation 19.6 into an integral equation. The general Green's function equation can be written eq18.15

$$(E - H_0)G = \delta(\mathbf{x} - \mathbf{x}') + V(\mathbf{x})G. \quad (19.8)$$

eq18.16

We can now use Green's second identity. We define an operator

$$L_0 \equiv E - H_0.$$

25 May p4

The operator L_0 is hermitian, since both E and H_0 are. Recall that Green's second identity is

$$\int (S^* L_0 u) = \int (u L_0 S),$$

where we now choose

$$\begin{aligned} S^* &= G(\mathbf{x}, \mathbf{x}'; E), \\ u &= G_0(\mathbf{x}, \mathbf{x}''; E), \end{aligned}$$

with the L_0 from above. We now have (using equation 19.8)

$$\begin{aligned} \int dx G(\mathbf{x}, \mathbf{x}'; E) \delta(\mathbf{x}' - \mathbf{x}'') &= \\ \int dx G_0(\mathbf{x}, \mathbf{x}''; E) [\delta(\mathbf{x} - \mathbf{x}') + V(\mathbf{x})G(\mathbf{x}, \mathbf{x}''; E)], \end{aligned}$$

so

$$G(\mathbf{x}'', \mathbf{x}') = G_0(\mathbf{x}', \mathbf{x}'') + \int dx G_0(\mathbf{x}, \mathbf{x}'') V(\mathbf{x}) G(\mathbf{x}, \mathbf{x}').$$

We can use the fact that G is symmetric (see equation 19.7) to write

$$G(\mathbf{x}'', \mathbf{x}'; E) = G_0(\mathbf{x}'', \mathbf{x}'; E) + \int dx G_0(\mathbf{x}'', \mathbf{x}; E) V(\mathbf{x}) G(\mathbf{x}, \mathbf{x}'; E).$$

So, for $\mathbf{x}'' \rightarrow \mathbf{x}$ and $\mathbf{x} \rightarrow \mathbf{x}_1$, we have

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) G(\mathbf{x}_1, \mathbf{x}'; E). \quad (19.9)$$

eq18.23
pr:FundInt1

This is called the fundamental integral equation of scattering. This integral is completely equivalent to equation 19.6. We now describe a way to write equation 19.9 diagrammatically. We establish the following correspondences.

$$G(\mathbf{x}, \mathbf{x}'; E) = \begin{array}{c} \xleftarrow{\quad} \bigcirc G \xleftarrow{\quad} \\ \mathbf{x} \qquad \qquad \mathbf{x}' \end{array}$$

$$G_0(\mathbf{x}, \mathbf{x}'; E) = \begin{array}{c} \xleftarrow{\quad} \xleftarrow{\quad} \\ \mathbf{x} \qquad \qquad \mathbf{x}' \end{array}$$

$$\int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) G(\mathbf{x}_1, \mathbf{x}'; E) = \begin{array}{c} \xleftarrow{\quad} \bullet \xleftarrow{\quad} \bigcirc G \xleftarrow{\quad} \\ \mathbf{x} \qquad \mathbf{x}_1 \qquad \qquad \mathbf{x}' \end{array}$$

Thus a line indicates a free Green's function. A dot indicates a potential, and a G in a circle represents the Green's function in the presence of the potential. The point \mathbf{x}_1 represents the position of the last interaction. Thus equation 19.9 can be written

25 May p5

$$\begin{array}{c} \xleftarrow{\quad} \bigcirc G \xleftarrow{\quad} \\ \mathbf{x} \qquad \qquad \mathbf{x}' \end{array} = \begin{array}{c} \xleftarrow{\quad} \xleftarrow{\quad} \\ \mathbf{x} \qquad \qquad \mathbf{x}' \end{array} + \begin{array}{c} \xleftarrow{\quad} \bullet \xleftarrow{\quad} \bigcirc G \xleftarrow{\quad} \\ \mathbf{x} \qquad \mathbf{x}_1 \qquad \qquad \mathbf{x}' \end{array}$$

The arrowheads indicate the line of causality. This helps us to remember the ordering of \mathbf{x}' , \mathbf{x}_1 and \mathbf{x} .

19.2 Formal Scattering Theory

Now we want to derive this equation again more formally. We will use the operator formalism, which we now introduce. The free Green's pr:OpForm1 function equation is

$$\left[E + \frac{\hbar^2}{2m} \nabla^2 \right] G_0(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}'),$$

where the right hand side is just the identity matrix,

$$\langle \mathbf{x} | 1 | \mathbf{x}' \rangle = \delta(\mathbf{x} - \mathbf{x}'),$$

and $(1)_{ij} = \delta_{ij}$. We also write

$$G_0(\mathbf{x}, \mathbf{x}'; E) \left[E + \frac{\hbar^2}{2m} \nabla'^2 \right] = \delta(\mathbf{x} - \mathbf{x}'),$$

where the operator ∇'^2 operates to the right. From these equations we can write symbolically

$$[E - H_0]G_0 = 1 \quad (19.10)$$

eq18.25

and

$$G_0 = 1[E - H_0]. \quad (19.11)$$

eq18.26

This uses the symmetry of G_0 operating on \mathbf{x} or \mathbf{x}' . Thus our manipulations are essentially based on hermiticity. Because G is also symmetric, we may also write

$$[E - H]G = 1,$$

and

$$G[E - H] = 1.$$

25 May p6

We now want to rederive equation 19.9. We write

$$[E - H]G = 1$$

and

$$[E - H_0 - V]G = 1.$$

We now multiply on the left by G_0 to get

$$G_0[E - H_0 - V]G = G_0 \cdot 1 = G_0.$$

With the aid of equation 19.11 this becomes

$$G - G_0VG = G_0,$$

or

$$G = G_0 + G_0VG, \quad (19.12)$$

where the term G_0VG symbolizes matrix multiplication, which is thus as integral. This is equivalent to equation 19.9, which is what we wanted to derive.

Note that

$$\langle \mathbf{x} | G | \mathbf{x}' \rangle = G(\mathbf{x}, \mathbf{x}')$$

and

$$\langle \mathbf{x} | V | \mathbf{x}' \rangle = V(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}').$$

19.2.1 A short digression on operators

If an integral of the form

$$C(\mathbf{x}_1, \mathbf{x}_2) = \int d\mathbf{x}' A(\mathbf{x}_1, \mathbf{x}') B\mathbf{x}', \mathbf{x}_2)$$

were written as a discrete sum, we would let $\mathbf{x}_1 \rightarrow i$, $\mathbf{x}_2 \rightarrow j$, and $\mathbf{x}' \rightarrow k$. We could then express it as

$$C_{ij} = \sum_k A_{ik} B_{kj}.$$

But now A , B , and C are just matrices, so we can express C as a matrix product $C = AB$. This can also be viewed as an operator equation. Quantum mechanically, this can be represented as a product of expectation values, either for a discrete spectrum,

$$\langle i | C | j \rangle = \sum_k \langle i | A | k \rangle \langle k | B | j \rangle,$$

or for a continuous spectrum,

$$\langle \mathbf{x}_1 | C | \mathbf{x}_2 \rangle = \int d\mathbf{x}' \langle \mathbf{x}_1 | A | \mathbf{x}' \rangle \langle \mathbf{x}' | B | \mathbf{x}_2 \rangle.$$

We now show that the form of the fundamental integral of scattering expressed in equation 19.12 is equivalent to that in equation 19.9. If we reexpress equation 19.12 in terms of expectation values, we have

$$\langle \mathbf{x} | G | \mathbf{x}' \rangle = \langle \mathbf{x} | G_0 | \mathbf{x}' \rangle + \langle \mathbf{x} | G_0 V G | \mathbf{x}' \rangle.$$

By comparing with equation 19.10, we see that the last term can be written as

$$\begin{aligned}
 \langle \mathbf{x} | G_0 V G | \mathbf{x}' \rangle &= \int d\mathbf{x}_1 d\mathbf{x}_2 \langle \mathbf{x} | G_0 | \mathbf{x}_1 \rangle \langle \mathbf{x}_1 | V | \mathbf{x}_2 \rangle \langle \mathbf{x}_2 | G | \mathbf{x}' \rangle \\
 &= \int d\mathbf{x}_1 d\mathbf{x}_2 \langle \mathbf{x} | G_0 | \mathbf{x}_1 \rangle V(\mathbf{x}_1) \delta(\mathbf{x}_1 - \mathbf{x}_2) \langle \mathbf{x}_2 | G | \mathbf{x}' \rangle \\
 &= \int d\mathbf{x}_1 \langle \mathbf{x} | G_0 | \mathbf{x}_1 \rangle V(\mathbf{x}_1) \langle \mathbf{x}_1 | G | \mathbf{x}' \rangle.
 \end{aligned}$$

So by identifying

$$\begin{aligned}
 \langle \mathbf{x} | G | \mathbf{x}' \rangle &= G(\mathbf{x}, \mathbf{x}') \\
 \langle \mathbf{x} | G_0 | \mathbf{x}' \rangle &= G_0(\mathbf{x}, \mathbf{x}')
 \end{aligned}$$

we have our final result, identical to equation 19.9,

$$G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1) V(\mathbf{x}_1) G(\mathbf{x}_1, \mathbf{x}')$$

27 May p1

which we obtained using equation 19.12.

19.3 Summary of Operator Method

We started with

$$H = H_0 + V$$

and the algebraic formulas

$$(E - H)G = 1,$$

$$(E_0 - H)G_0 = 1.$$

We then found that G satisfies the integral equation

$$G + G_0 + G_0 V G.$$

By noting that $G(E - H) = 1$, we also got

$$G = G_0 + G V G_0.$$

27 May p2

19.3.1 Derivation of $G = (E - H)^{-1}$

The trick is to multiply by G_0 . Thus

$$G(E - H_0 - V) = 1 \cdot G_0,$$

and so

$$G - GVG_0 = G_0.$$

Operators are nothing more than matrices. By inverting the equation, we get

$$G = \frac{1}{E - H}.$$

So G is the inverse operator of $[E - H]$. In this context it is useful to define G in terms of its matrix elements:

$$G(\mathbf{x}, \mathbf{x}'; E) \equiv \langle \mathbf{x} | G(E) | \mathbf{x}' \rangle$$

where $\text{Im } E = 0$. This arithmetic summarizes the arithmetic of Greens Second Identity. We also found that the Green's function solves the following integral equation

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int dx G_0(\mathbf{x}, \mathbf{x}_1; E) V(x_1) G(\mathbf{x}_1, \mathbf{x}'; E).$$

We were able to express this graphically as well. The other equation gives

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int dx G(\mathbf{x}, \mathbf{x}_1; E) V(x_1) G_0(\mathbf{x}_1, \mathbf{x}'; E).$$

19.3.2 Born Approximation

Suppose the $V(x)$ is small. Then in the first approximation $G \sim G_0$. We originally used this to calculate the scattering amplitude f . We now use perturbation methods to obtain a power series in V .

19.4 Physical Interest

We now want to look at $E = E + i\varepsilon$. We place the source point at $x' \rightarrow -r'\hat{z}$, where $r' \rightarrow \infty$. We can then write the free space Green's function as

$$G_0(\mathbf{x}, \mathbf{x}'; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} e^{i\mathbf{k}\cdot\mathbf{x}}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}.$$

In this limit the full Green's function becomes, for the fundamental integral equation of scattering, equation 19.9,

$$\lim_{\substack{\mathbf{x} \rightarrow r'\hat{l}_z \\ r' \rightarrow \infty}} G(\mathbf{x}, \mathbf{x}'; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{kr'}}{r'} \left[e^{i\mathbf{k}\cdot\mathbf{x}} + \int dx_1 G(\mathbf{x}, \mathbf{x}_1; E) V(x_1) e^{i\mathbf{k}\cdot\mathbf{x}_1} \right]. \quad (19.13)$$

eq18.13a

Note that since

$$H = -\frac{\hbar^2}{2m} \nabla^2$$

we have

$$[E - H_0]e^{i\mathbf{k}\cdot\mathbf{x}} = 0.$$

27 May p5

We define

$$\Psi_{\mathbf{k}}^{(+)}(x) \equiv e^{i\mathbf{k}\cdot\mathbf{x}} + \int dx_1 G(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) e^{i\mathbf{k}\cdot\mathbf{x}_1}. \quad (19.14)$$

twosedblst

Then what we have shown is

$$\lim_{r' \rightarrow \infty} G(\mathbf{x}, \mathbf{x}'; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \Psi_l^{(+)}(x)$$

where $[E - H]\Psi_k^+ = 0$, and Ψ_k^+ satisfies outgoing wave boundary condition for scattering. We can get $\Psi_{\mathbf{k}}^{(+)}$ to any order in perturbation since we have an explicit expression for it and G .

Now consider the case in which $r' \rightarrow \infty$ with

1. $G = G_0 + G_0 V G$
2. $G = G_0 + G V G_0$

Case 2 implies that

$$G = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \Psi_n^{(+)}(\mathbf{x}) \quad \text{as } r' \rightarrow \infty. \quad (19.15)$$

By inserting Eq. (19.15) into 1., we get

eq:twoseva

$$-\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \Psi_N^{(+)}(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \left[e^{i\mathbf{k}\cdot\mathbf{x}} + \int G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}_1) \right].$$

This implies that $\Psi_{\mathbf{k}}^{(+)}(\mathbf{x})$ satisfies

27 May p6

$$\Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \int G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}_1). \quad (19.16)$$

This is the fundamental integral expression for Ψ . Compare with Eq. *twosevst* (19.14).

We now prove that $\Psi_{\mathbf{k}}^{(+)}$ satisfies scattering equation with the scattering condition. We use the form of $\Psi^{(+)}$ in Eq. (19.14).

$$\begin{aligned} (E - H) \Psi_{\mathbf{k}}^{(+)} &= (E - H_0 - V) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\quad + \int d\mathbf{x}_1 (E - H_0 - V)_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) e^{i\mathbf{k}\cdot\mathbf{x}_1} \\ &= -V(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} + \int d\mathbf{x}_1 \delta(\mathbf{x} - \mathbf{x}') V(\mathbf{x}_1) e^{i\mathbf{k}\cdot\mathbf{x}_1} \\ &= -V(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} + V(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= 0. \end{aligned}$$

27 May p7

19.4.1 Satisfying the Scattering Condition

We use the form of equation (19.16) to prove that it does satisfy the scattering condition. Let $\mathbf{x} = \hat{e}_r r$ where $r \rightarrow \infty$. We use the result

$$\lim_{\substack{\mathbf{x} = r\hat{e}_r \\ r \rightarrow \infty}} G_0(\mathbf{x}, \mathbf{x}_1; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} e^{-i\mathbf{k}_f \cdot \mathbf{x}_1}$$

where $\mathbf{k}_f = k\hat{e}_r$. There is a minus in the exponent since we are taking the limit as \mathbf{x} approaches $\hat{e}_r\infty$ rather than $-\hat{e}_r\infty$ as in equation (19.15). Thus the limiting case is

$$\Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r} \left[-\frac{m}{2\pi\hbar^2} \int d\mathbf{x}_1 e^{-i\mathbf{k}_f\cdot\mathbf{x}_1} V(\mathbf{x}_1) \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}_1) \right].$$

We further define

$$\lim_{\substack{\mathbf{x}=r\hat{e}_r \\ r\rightarrow\infty}} \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) \equiv e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r} f(\mathbf{k}, \mathbf{k}_f; E)$$

where f is the scattering amplitude to scatter a particle of incident wave \mathbf{k} to outgoing \mathbf{k}_f with energy E . This is the outgoing wave boundary condition.

27 May p8

19.5 Physical Interpretation

We defined the wave function $\Psi_{\mathbf{k}}^{(+)}(\mathbf{x})$ using equation 19.13 whose components have the following interpretation.

$$\begin{aligned} \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) &= e^{i\mathbf{k}\cdot\mathbf{x}} + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}_1) \\ &\equiv \Psi_{\text{incident}}(\mathbf{x}) + \Psi_{\text{scattered}}(\mathbf{x}) \end{aligned}$$

where

$$\lim_{|\mathbf{x}|=r\rightarrow\infty} \Psi_s(\mathbf{x}) = \frac{e^{ikr}}{r} f(\mathbf{k}, \mathbf{k}_f; E).$$

The physical interpretation of this is shown graphically as follows.

$$\Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) = \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} + \begin{array}{c} \text{(((((\\ (((((\\ (((((\\ \bullet \\ ((((((\\ ((((((\\ \text{)))))) \\ \text{)))))) \end{array} \quad \begin{array}{c} \Psi_{\text{incident}}(\mathbf{x}) \quad \Psi_{\text{scattered}}(\mathbf{x}) \end{array}$$

19.6 Probability Amplitude

The differential cross section is given by

27 May p9
pr:differsec1

$$\frac{d\sigma}{d\Omega}(\mathbf{k} \rightarrow \mathbf{k}_f) = |f(\mathbf{k}, \mathbf{k}_f; E)|^2$$

where f is the scattering amplitude for scattering with initial momentum $\mathbf{p} = \hbar\mathbf{k}$ and final amplitude $\mathbf{p}_f = \hbar\mathbf{k}_f$ from a potential $V(\mathbf{x})$. The scattering amplitude f is sometimes written

$$f(\mathbf{k}, \mathbf{k}_f; E) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{k}_f | V | \Psi_{\mathbf{k}}^{(+)} \rangle.$$

19.7 Review

1 Jun p1

We have obtained the integral equation for the Green's function,

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int dx G_0(\mathbf{x}, \mathbf{x}_1; E) V(x_1) G(\mathbf{x}_1, \mathbf{x}'; E).$$

For the case of a distant source we have seen

$$\lim_{\mathbf{x}' \rightarrow -\infty \hat{z}} G(\mathbf{x}, \mathbf{x}'; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr'}}{r'} \Psi_E^{(+)}(\mathbf{x}),$$

where

$$\Psi_E^{(+)}(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) \Psi_E^{(+)}(\mathbf{x}_1).$$

The first term is a plane wave. The integral represents a distorted wave. Note that $\Psi_E^{(+)}(\mathbf{x})$ automatically satisfies the outgoing wave boundary condition. (This is the advantage of the integral equation approach over the differential equation approach.) To verify this, we took the limit $\mathbf{x} \rightarrow \infty$. We also obtained

$$\Psi_E^{(+)}(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + \int d\mathbf{x}_1 G(\mathbf{x}_1, \mathbf{x}'; E) V(\mathbf{x}_1) e^{i\mathbf{k} \cdot \mathbf{x}_1}.$$

We let $E \rightarrow E + i\epsilon$ to get a scattering solution,

$$G_0(\mathbf{x}, \mathbf{x}'; E + i\epsilon) = \frac{e^{i\mathbf{k} \cdot |\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \left(-\frac{m}{2\pi\hbar^2} \right),$$

where $|\mathbf{k}| = \sqrt{2mE/\hbar^2}$. We also have shown that $\Psi_E^{(+)}(\mathbf{x})$ satisfies

$$[E - H] \Psi_E^{(+)}(\mathbf{x}) = 0.$$

The wave function $\Psi_E^{(+)}(\mathbf{x})$ can also be written in the form

$$\Psi_E^{(+)}(\mathbf{x}) \xrightarrow{\mathbf{x} \rightarrow \infty} e^{i\mathbf{k} \cdot \mathbf{x}} + f(\mathbf{k}, \mathbf{k}_1; E) \frac{e^{ikr}}{r},$$

where we have obtained the following unique expression for f ,

$$f(\mathbf{k}, \mathbf{k}_1; E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{x} e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(\mathbf{x}) \Psi_E^{(+)}(\mathbf{x}),$$

where the integral represents a distorted wave. In particular, the term $e^{-i\mathbf{k}_f \cdot \mathbf{x}}$ is a free wave with the final momentum, $V(\mathbf{x})$ is the interaction potential, and $\Psi_E^{(+)}(\mathbf{x})$ is the distorted wave. So the integral expression for f is the overlap of Ψ and V with the outgoing final wave. Note that we have made no use of spherical symmetry. All \mathbf{x} contribute, so we still need short distance behavior even for far distance results. The differential cross section can be written in terms of f as

$$\left. \frac{d\sigma}{d\Omega} \right|_{\mathbf{k} \rightarrow \mathbf{k}_f} = |f|^2.$$

19.8 The Born Approximation

pr:BornAp1

We now study a particular approximation technique to evaluate $\Psi_E^{(+)}(\mathbf{x})$ in

$$\Psi_E^{(+)}(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}'; E) V(\mathbf{x}_1) \Psi_E^{(+)}(\mathbf{x}_1).$$

We assume that the potential is weak so that the distortion, as represented by the integral, is small. The condition that the distortion is small is

$$\text{small distortion} \quad \Longleftrightarrow \quad |\Psi_E^{(+)}(\mathbf{x}) - e^{i\mathbf{k} \cdot \mathbf{x}}| \ll 1.$$

In this case the potential must be sufficiently small, such that

$$\int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}'; E) V(\mathbf{x}_1) \Psi_E^{(+)}(\mathbf{x}_1) \ll 1.$$

We now introduce the short hand of representing this integral by V_B , the Born parameter:

$$V_B \equiv \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}'; E) V(\mathbf{x}_1) \Psi_E^{(+)}(\mathbf{x}_1) \ll 1.$$

For $V_B \ll 1$ we may let $\Psi_E^{(+)}(\mathbf{x}_1)$ be replaced by $e^{i\mathbf{k}\cdot\mathbf{x}}$ in $f(\mathbf{k}, \mathbf{k}_f; E)$. In this case f becomes $f_{\text{Born}}(\mathbf{k}, \mathbf{k}_f; E)$, defined by

$$f(\mathbf{k}, \mathbf{k}_f; E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{x} e^{-i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}_f)} V(\mathbf{x}).$$

In this approximation the cross section becomes

$$\frac{d\sigma}{d\Omega} \xrightarrow{\text{Born}} \frac{d\sigma_B}{d\Omega} = |f_B|^2.$$

This is called the first Born approximation. This approximation is valid in certain high energy physics domains.

We now introduce the matrix notation

$$\int d\mathbf{x} e^{-i\mathbf{x}\cdot\mathbf{k}_f} V(\mathbf{x}) \Psi_{\mathbf{k}}^{(+)}(\mathbf{x}) \equiv \langle \mathbf{x}_f | V | \Psi_{\mathbf{k}}^{(+)} \rangle.$$

So in terms of this matrix element the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}, \mathbf{k}_f; E)|^2,$$

where the scattering amplitude is given by

$$f(\mathbf{k}, \mathbf{k}_f; E) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{x}_f | V | \Psi_{\mathbf{k}}^{(+)} \rangle.$$

We also define the “wave number” transfer \mathbf{q} ,

$$\mathbf{q} = \mathbf{k}_f - \mathbf{k} = (\mathbf{p}_f - \mathbf{p}_i)/\hbar.$$

Thus \mathbf{q} is the same as (momentum transfer)/ \hbar . This allows us to write

$$f_B = -\frac{m}{2\pi\hbar^2} \tilde{V}(\mathbf{q}),$$

where the fourier transformed potential, $\tilde{V}(\mathbf{q})$, is given by

$$\tilde{V}(\mathbf{q}) = \int d\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}).$$

So in the first Born approximation, f_B depends only on \mathbf{q} .

Suppose that $\mathbf{q} \rightarrow 0$. In this case the potential simplifies to

$$\tilde{V}(\mathbf{q}) \xrightarrow{\mathbf{q} \rightarrow 0} \int d\mathbf{x} V(\mathbf{x}).$$

So the first Born approximation just gives us the f dependence on the average of the potential. Notice that the first Born approximation loses the imaginary part of $f(\mathbf{k}, \mathbf{k}_f; E)$ for f_B in R , the real numbers.

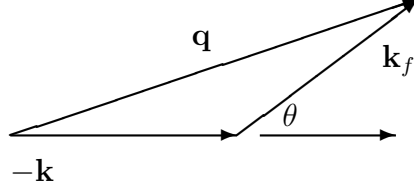


Figure 19.1: Geometry of the scattered wave vectors.

19.8.1 Geometry

fig18a

The relationship between \mathbf{k} , \mathbf{k}_f , \mathbf{q} and θ is shown in figure 19.1. For the special case of elastic scattering we have

$$k_f^2 = k^2 = 2mE/\hbar^2 \quad (\text{elastic scattering}).$$

In this case q^2 is given by

$$\begin{aligned} q^2 &= (\mathbf{k}_f - \mathbf{k}) \cdot (\mathbf{k}_f - \mathbf{k}) \\ &= 2k^2 - 2k^2 \cos \theta \\ &= 2k^2(1 - \cos \theta) \\ &= 4k^2 \sin^2(\theta/2). \end{aligned}$$

Thus we have $q = 2k \sin(\theta/2)$. We thus know that q will be small for either $k \rightarrow 0$ (the low energy limit) or $\sin(\theta/2) \rightarrow 0$ (forward scattering).

19.8.2 Spherically Symmetric Case

In this case the potential $V(\mathbf{x})$ is replaced by $V(r)$. We choose the z -axis along \mathbf{q} and use spherical coordinates. The fourier transform of the potential then becomes

$$\begin{aligned} \tilde{V}(\mathbf{q}) &= \int r^2 dr d\phi d(\cos \theta) e^{-iqr \cos \theta} V(r) \\ &= 2\pi \int_0^\infty r^2 dr V(r) \int_{-1}^1 d(\cos \theta) e^{-iqr \cos \theta} \\ &= \frac{4\pi}{q} \int_0^\infty r dr V(r) \sin qr. \end{aligned}$$

This is a 1-dimensional fourier sine transform.

19.8.3 Coulomb Case

We now choose a specific $V(r)$ so that we can do the integral. We choose the shielded Coulomb potential,

$$V(r) = \frac{V_0}{r} e^{-r/a}.$$

In the problem set we use α instead of V_0 . The parameter α characterizes the charge. The fourier sine transform of this potential is

$$\tilde{V}(\mathbf{q}) = \frac{4\pi\alpha}{q} \int_0^\infty dr \sin r q e^{-r/a},$$

and the differential cross section is then

$$\begin{aligned} \left. \frac{d\sigma}{d\Omega} \right|_{\text{Born}} &= |f_B|^2 \\ &= \left| -\frac{m}{2\pi\hbar^2} \tilde{V}(\mathbf{q}) \right|^2 \\ &= 4a^2 \left(\frac{\alpha m a}{\hbar^2} \right)^2 \left(\frac{1}{q^2 a^2 + 1} \right)^2. \end{aligned}$$

This is the shielded Coulomb scattering differential cross section in the first Born approximation, where $q = 2k \sin(\theta/2)$. Notice that as $\alpha \rightarrow \infty$ this reduces to Rutherford scattering, which is a lucky accident.

We now look at characteristics of the differential cross section we have obtained. Most of the cross section contribution comes from $qa = 2k \sin(\theta/2) \ll 1$. Now if $ka \gg 1$, then we must require $\theta \ll 1$, which means that we can use the small angle approximation. In this case our dominant cross section condition becomes $qa \approx 2ka(\theta/2) \ll 1$, or $\theta \ll 1/ka$. This gives a quantitative estimation of how strongly forward peaked the scattered wave is. The condition $ka \gg 1$ corresponds to the small λ , or high energy, limit. In this case the wavelength is much smaller than the particle, which means that most of the scattering will be in the forward direction. We can see how good the first Born approximation is by evaluating the Born parameter in this limit. We find

$$V_B = V_0 \frac{ma^2}{k^2} \frac{1}{ka} \ll 1.$$

In this equation V_0 is the strength of the potential and a is the range of the potential. Notice that $ka \gg 1$ can make $V_B \ll 1$ even if V_0 is large. Thus we have a dimensionless measure of the strength of the potential.

19.9 Scattering Approximation

We now want to look at the perturbation expansion for the differential cross section,

$$\left. \frac{d\sigma}{d\Omega} \right|_{\mathbf{k} \rightarrow \mathbf{k}_f} = |f(\mathbf{k}, \mathbf{k}_f)|^2,$$

where the scattering amplitude $f(\mathbf{k}, \mathbf{k}_f)$ is

$$f(\mathbf{k}, \mathbf{k}_1; E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{x} e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(\mathbf{x}) \Psi_E^{(+)}(\mathbf{x}), \quad (19.17)$$

eq18b1

where the incident wave function is

$$\Psi_E^{(+)}(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} + \int d\mathbf{x}_1 G(\mathbf{x}_1, \mathbf{x}'; E) V(\mathbf{x}_1) e^{i\mathbf{k} \cdot \mathbf{x}_1}. \quad (19.18)$$

eq18b2

$\Psi_E^{(+)}(\mathbf{x})$ satisfies the outgoing wave condition. By combining equation 19.17 into 19.18 we obtain

$$\begin{aligned} f(\mathbf{k}, \mathbf{k}_1; E) = & -\frac{m}{2\pi\hbar^2} \left\{ \int d\mathbf{x} e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right. \\ & \left. \int d\mathbf{x} d\mathbf{x}' e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(\mathbf{x}) G(\mathbf{x}, \mathbf{x}'; E) V(\mathbf{x}') e^{i\mathbf{k} \cdot \mathbf{x}'} \right\}. \end{aligned}$$

The first integral represents a single interaction, while the second integral represents two or more interactions. By introducing the transition operator, we can simplify the expression for the scattering amplitude,

$$f(\mathbf{k}, \mathbf{k}_1; E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{x} d\mathbf{x}' e^{-i\mathbf{k}_f \cdot \mathbf{x}} G(\mathbf{x}, \mathbf{x}'; E) e^{i\mathbf{k} \cdot \mathbf{x}'}.$$

pr:transOp1

We now define the transition operator T . In function notation it is

$$T(\mathbf{x}, \mathbf{x}'; E) \equiv V(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}') + V(\mathbf{x}) G(\mathbf{x}, \mathbf{x}'; E) V(\mathbf{x}').$$

In operator notation, we can rewrite this equation as

$$T = V + VGV.$$

Thus in matrix notation, our old equation for f ,

$$f(\mathbf{k}, \mathbf{k}_f; E) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{x}_f | V | \Psi_{\mathbf{k}}^{(+)} \rangle,$$

is replaced by

$$f(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi\hbar^2} \langle \mathbf{x}_f | T | \mathbf{k} \rangle,$$

Thus we now have two equivalent forms for expressing f .

In the first Born approximation we approximate $T = V$. T plays the role in the exact theory what V plays in the first Born approximation.

19.10 Perturbation Expansion

We now look at how the transition operator T can be used in diagrammatic perturbation theory. We make the following correspondences between terms in the formulas and the graphical counterparts (these are the “Feynman rules”):

pr:pertExp1

An incoming line:

 represents $e^{i\mathbf{k}\cdot\mathbf{x}}$.

An outgoing line:

 represents $e^{-i\mathbf{k}_f\cdot\mathbf{x}}$.

A vertex point:

 represents $V(\mathbf{x})$.

A free propagator:

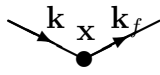
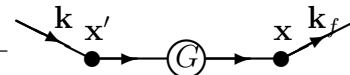
 represents $G_0(\mathbf{x}_2, \mathbf{x}_1)$.

A circled G :

 represents $G(\mathbf{x}, \mathbf{x}')$.

Thus we can write the transition operator matrix element as

$$\langle \mathbf{x}_f | T | \mathbf{k} \rangle = \text{Diagram 1} + \text{Diagram 2}$$

Diagram 1:  + Diagram 2: 

The first diagram represents the first Born approximation, which corresponds to a single scatterer. The second diagram represents two or

pr:FeynRul1

more scatterer, where the propagation occurs via any number of interactions through G .

The integral equation for the full Green's function,

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int d\mathbf{x}_1 G(\mathbf{x}, \mathbf{x}_1; E) V(\mathbf{x}_1) G(\mathbf{x}_1, \mathbf{x}'; E),$$

has the following symbolic representation:

$$\begin{array}{c} \mathbf{x}' \\ \rightarrow \textcircled{G} \rightarrow \mathbf{x} \end{array} = \begin{array}{c} \mathbf{x}' \quad \mathbf{x} \\ \rightarrow \quad \rightarrow \end{array} + \begin{array}{c} \mathbf{x} \quad \mathbf{x}_1 \\ \rightarrow \quad \bullet \rightarrow \end{array} \textcircled{G} \rightarrow \mathbf{x}$$

where \mathbf{x}' is the source point, \mathbf{x} is the field point, and \mathbf{x}_1 is one of the interaction points.

19.10.1 Perturbation Expansion

In matrix language the integral equation for the full Green's function is

$$G = G_0 + GV G_0,$$

which implies

$$G = G_0(1 - VG_0)^{-1}.$$

Thus the following geometric series gives the solution to the integral equation,

$$G = G_0(1 + VG_0 + (VG_0)(VG_0) + (VG_0)(VG_0)(VG_0) + \cdots).$$

In symbolic notation, this expansion corresponds to

$$\begin{array}{c} \mathbf{x}' \\ \rightarrow \textcircled{G} \rightarrow \mathbf{x} \end{array} = \begin{array}{c} \mathbf{x}' \quad \mathbf{x} \\ \rightarrow \quad \rightarrow \end{array} + \begin{array}{c} \mathbf{x}' \quad \mathbf{x}_1 \quad \mathbf{x} \\ \rightarrow \quad \bullet \rightarrow \quad \rightarrow \end{array} + \begin{array}{c} \mathbf{x}' \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x} \\ \rightarrow \quad \bullet \rightarrow \quad \bullet \rightarrow \quad \rightarrow \end{array} \\ + \begin{array}{c} \mathbf{x}' \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \mathbf{x} \\ \rightarrow \quad \bullet \rightarrow \quad \bullet \rightarrow \quad \bullet \rightarrow \quad \rightarrow \end{array} + \cdots.$$

We could also write the series expansion in integral notation. In this case the third order in V term, $(VG_0)(VG_0)(VG_0)$, is (writing right to left)

$$\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 G_0(\mathbf{x}, \mathbf{x}_3) V(\mathbf{x}_3) G_0(\mathbf{x}_3, \mathbf{x}_2) V(\mathbf{x}_2) G_0(\mathbf{x}_2, \mathbf{x}_1) V(\mathbf{x}_1) G_0(\mathbf{x}_1, \mathbf{x}'),$$

where, for example,

$$G_0(\mathbf{x}_3, \mathbf{x}_2) = -\frac{m}{2\pi\hbar^2} \frac{e^{i\mathbf{k}\cdot|\mathbf{x}_2-\mathbf{x}_3|}}{|\mathbf{x}_2-\mathbf{x}_3|}.$$

Think of these terms as multiply scattered terms.

Now we can use this series to get a perturbation expansion for the scattering amplitude f , that is, for the matrix element $\langle \mathbf{x}_f | T | \mathbf{k} \rangle$. In symbolic language it is

is this correct?

$$\begin{aligned} \langle \mathbf{x}_f | T | \mathbf{k} \rangle &= \text{diagram 1} + \text{diagram 2} \\ &= \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \dots \end{aligned}$$

The diagrams represent the perturbation expansion of the scattering amplitude. Diagram 1 shows a single vertex with incoming momentum \mathbf{k} and position \mathbf{x} , and outgoing momentum \mathbf{k}_f and position \mathbf{x} . Diagram 2 shows a vertex with incoming momentum \mathbf{k} and position \mathbf{x}' , followed by a propagator G , and then a vertex with outgoing momentum \mathbf{k}_f and position \mathbf{x} . Diagram 3 shows a vertex with incoming momentum \mathbf{k} and position \mathbf{x} , followed by a vertex with incoming momentum \mathbf{k} and position \mathbf{x}' , and then a vertex with outgoing momentum \mathbf{k}_f and position \mathbf{x} . Diagram 4 shows a vertex with incoming momentum \mathbf{k} and position \mathbf{x}' , followed by two vertices with positions \mathbf{x}_1 and \mathbf{x}_2 , and then a vertex with outgoing momentum \mathbf{k}_f and position \mathbf{x} . Diagram 5 shows a vertex with incoming momentum \mathbf{k} and position \mathbf{x}' , followed by three vertices with positions \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{x} , and then a vertex with outgoing momentum \mathbf{k}_f and position \mathbf{x} .

To convert this to integral language we note that, for example, the fourth Born approximation term is

$$\text{diagram 4}$$

In integral notation this is expressed as

$$\int d\mathbf{x} d\mathbf{x}' d\mathbf{x}_1 d\mathbf{x}_2 \left[e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(\mathbf{x}) G_0(\mathbf{x}, \mathbf{x}_2) V(\mathbf{x}_2) G_0(\mathbf{x}_2, \mathbf{x}_1) V(\mathbf{x}_1) G_0(\mathbf{x}_1, \mathbf{x}') e^{i\mathbf{k} \cdot \mathbf{x}'} \right].$$

We must integrate over all space since each of the interaction points may occur at any place.

19.10.2 Use of the T -Matrix

An alternative approach is to eliminate all direct reference to G without perturbation theory. We then obtain an integral equation for the transition matrix. By using

$$G = G_0(1 - VG_0)^{-1},$$

we have

$$VG = VG_0(1 - VG_0)^{-1},$$

so we can write the transition matrix as

$$\begin{aligned}
 T &= V + VG_0(1 - VG_0)^{-1}V \\
 &= [1 + VG_0(1 - VG_0)^{-1}]V \\
 &= [(1 - VG_0) + VG_0](1 - VG_0)^{-1}V \\
 &= (1 - VG_0)^{-1}V.
 \end{aligned}$$

This provides us with a new solution for T :

$$T = (1 - VG_0)^{-1}V.$$

We can write this as an integral equation, which would have the operator form

$$(1 - VG_0)T = V,$$

or

$$T = V + VG_0T.$$

This gives us another Lippman/Schwinger equation. Notice that $T = (1 - VG_0)^{-1}V$ may be expanded in a power series in V just as was the previous expression for G .

19.11 Summary

1. The fundamental integral equation of scattering is

$$G(\mathbf{x}, \mathbf{x}'; E) = G_0(\mathbf{x}, \mathbf{x}'; E) + \int d\mathbf{x}_1 G_0(\mathbf{x}, \mathbf{x}_1; E)V(\mathbf{x}_1)G(\mathbf{x}_1, \mathbf{x}'; E).$$

19.12 References

See [Neyfeh, p360ff] for perturbation theory.

Appendix A

Symbols Used

$\langle S, u \rangle$ the brackets denote an inner product, 13.

$*$ as a superscript, represents complex conjugation, 13.

∇ nabla, the differential operator in an arbitrary number of dimensions, 7.

A_1, A_2 constants used in determining the Green's function, 28.

a the horizontal displacement between mass points on a string; an arbitrary position on the string, 2, the left endpoint of a string, 6.

a_1, a_2 constants used in discussion of superposition, 23.

B_1, B_2 constants used in determining the Green's function, 28.

b the right endpoint of a string 6.

$b(x)$ width of a water channel, 108.

C a constant used in determining the Green's function, 29.

c left endpoint used in the discussion of the δ -function, 24; constant characterizing velocity, 39, 45.

D a constant used in determining the Green's function, 29.

d the differential operator; right endpoint used in the discussion of the δ -function, 24.

Δp change in momentum, 87.

Δu_i the transverse distance between adjacent points ($u_i - u_{i-1}$) on a discrete string, 4.

Δx the longitudinal distance between adjacent points on a discrete string, 4.

$\delta(x - x')$ the delta function, 24, 129, 161.

δ_{mn} the Kroneker delta function, 36.

E energy, 74, 143.

$e = 2.71 \dots$.

ϵ a small distance along the string, 27.

$F(x, t)$ the external force on a continuous string, 1.

F_{cd} the force over the interval $[c, d]$, used in the discussion of the δ -function, 24.

F_i^{elastic} the elastic force on the i th mass point of a discrete string, 3.

F_i^{ext} the external force on the i th mass point of a discrete string, 3.

$F_{iy}^{\tau_i}$ the transverse force at the i th mass point on a string due to tension, 3.

F_{tot} the total force on the i th mass point of the string.

$f(x)$ is the external force density divided by the mass density at position x , 4.

$f(x')$ a finite term used in discussion of asymptotic Green's function, 42.

f_1, f_2 force terms used in discussion of superposition, 23.

$f(\theta, k)$ the scattering amplitude for a field observer from an incident plane wave, 214.

$\tilde{f}(\theta, r', k)$ scattering amplitude, 214.

$G(x, x_k; \omega^2)$ the Green's function for the Helmholtz equation, 26.

G_A the advances Green's function, 87.

G_S the scattered part of the steady state Green's function, 184.

G_R the retarded Green's function, 86.

$G_m(r, r'; \lambda)$ reduced Green's function, 132.

\tilde{G} the Fourier transform of the Green's function, 88, the Laplace transform of the Green's function, 147.

$g_n(x, x')$ asymptotic coefficient for Green's function near an eigen value, 41.

γ angular difference between \mathbf{x} and \mathbf{x}' used in scattering discussion, 185.

H the Hamiltonian, 195

$H_m^{(1)}(x)$, $H_m^{(2)}(x)$ the first and second Hankel functions, 79.

$h(x)$ equilibrium height of a surface wave, 108.

$h_l(x)$ the spherical Hankel function, 178.

$h_a(t)$ the effective force exerted by the string: F_a/τ_a , 6.

$h_S(t)$ same as $h_a(t)$, generalized for both endpoints, 8.

\hbar the reduced Plank's constant, 74.

$I(\omega)$ a general integral used in discussion of method of steepest descent, 265.

I_m another Bessel function, 80.

i the index of mass points on a string, 2.

\hat{i} unit vector in the x -direction, 128.

J Jacobian function, 128.

$\mathbf{j}(\mathbf{r})$ the quantum mechanical current density, 227.

j_{inc} the incident flux, 227.

$j_l(x)$ the spherical Bessel function, 178.

\mathbf{j}_n heat current, 144.

\hat{j} unit vector in the y -direction, 128.

K_m another Bessel function, 80.

k the wave number, 38.

k^2 a short hand for V/τ used in infinite string problem, 63.

k_i the spring constant at the i th mass point, 3.

κ the thermal diffusivity, 151.

κ_a the effective spring constant exerted by the string at endpoint a :
 k_a/τ_a , 6.

L_0 linear operator, 5.

$L_{\theta\varphi}$ centrifugal linear operator, 162.

\mathbf{L} the angular momentum vector, 207.

l dimension of length, 3.

\hat{l} the direction along the string in the positive x direction, 7.

λ an arbitrary complex number representing the square of the frequency continued into the complex plane, 27; wavelength of surface waves, 108.

- λ_n n th eigen value for the normal mode problem, 37.
- $\lambda_n^{(m)}$ the n th eigenvalue of the reduced operator $L_0^{(\mu_m)}$, 133.
- m dimension of mass, 3.
- m_i the mass of the particle at point i on the discrete string, 2.
- μ_m eigenvalues for circular eigenfunctions, 131.
- N the number of mass particles on the discrete string, 2; the number of particles intercepted in a scattering experiment, 227.
- $n_l(x)$ the spherical Neumann function, 178.
- \hat{n} the outward normal, 7.
- Ω solid angle, 161.
- ω angular frequency, 9.
- ω_n the natural frequency of the n th normal mode, 32.
- p momentum, 74, 207.
- Φ solution of the Klein Gordon equation, 75; total response due to a plane wave scattering on an obstacle, 185.
- Φ_0 incident plane wave used in scattering discussion, 185.
- ϕ angular coordinate, 128, 160.
- $\phi_n(x_i, t)$ the normal modes, 38.
- ψ quantum mechanical wave function, 195
- $R(r)$ function used to obtain Bessel's equation, 178.
- Re** take real value of whatever term immediately follows.
- r radial coordinate, 128.

S the “surface” (i.e., endpoints) of a one dimensional string, 7; an arbitrary function used in the derivation of the Green’s identities, 13.

$S(x)$ cross sectional area of a surface wave, 108.

σ the cross section, 227.

$\sigma(x)$ the mass density of the string at position x 4.

$T(\mathbf{x}, \mathbf{x}'; E)$ transition operator, 298.

t time, dimension 3, variable, 3.

τ_i the tension on the segment between the $(i-1)$ th and i th mass points on a string, 2.

Θ parameter in RBC for the heat equation, 251.

θ the angle of the string between mass points on a discrete string, 3; angle in parameterization of complex plane, 63.

$u(x, t)$ transverse displacement of string, 5; displacement of a surface wave from equilibrium height, 108.

$u_0(x)$ an arbitrary function used in the derivation of the Green’s identities, 13.

$u_0(x)$ value of the transverse amplitude at $t = 0$, 8.

$u_0(\mathbf{x}, \omega)$ steady state in free space due to a point source, 184.

$u_1(x)$ value of the derivative of the transverse amplitude at $t = 0$, 8.

u_1, u_2 functions used in discussion of superposition, 23.

u_1 solution of the homogeneous fixed string problem, 46.

u_i the vertical displacement of the i th mass particle on a string, 2.

$u_n^{(m)}(r)$ the n th eigenfunction of $L_0^{(\mu_m)}$, 133.

$u_l^m(x)$ the normalized θ -part of the spherical harmonic, 164.

- u_{scat} the scattered part of the steady state response, 198.
- \bar{u}_1 modified solution of the homogeneous fixed string problem, 47.
- $V(x)$ the coefficient of elasticity of the string at position x , 1.
- V_{eff} the effective potential, 206.
- $W(u_1, u_2)$ the Wronskian, 30.
- X_l coefficient of the scattered part of the wave relative to the incident part, 187, 219.
- x continuous position variable, 4.
- $x_<$ the lower of the position point and source point, 30.
- $x_>$ the higher of the position point and source point, 30.
- x' the location of the δ -function disturbance, 24.
- x_i discrete position variable, 4.
- x_k the location of the δ -function disturbance, 26.
- $Y_l^m(\theta, \varphi)$ the spherical harmonics, 164.
- $z(x)$ hight of a surface wave, 108.

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